

# **An Introduction to The Colored Jones Polynomial**

**LSU Math Geaux Seminar**

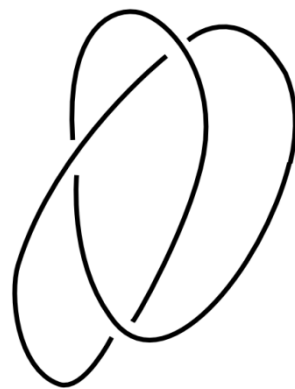
**Mustafa Hajij**

## Basic Definitions

**Definition** *A knot is a smooth embedding of  $S^1$  in  $\mathbb{R}^3$ .*

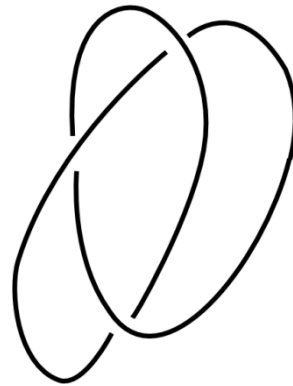
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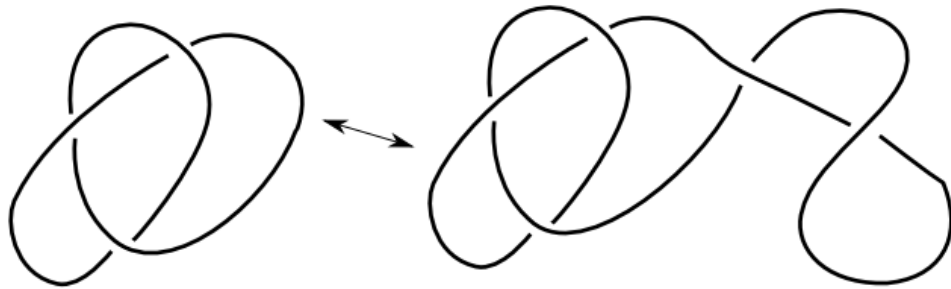
**Definition** *A knot is a smooth embedding of  $S^1$  in  $\mathbb{R}^3$ .*



A link is a smooth embedding of disjoint union of finite number of  $S^1$  in  $\mathbb{R}^3$ .

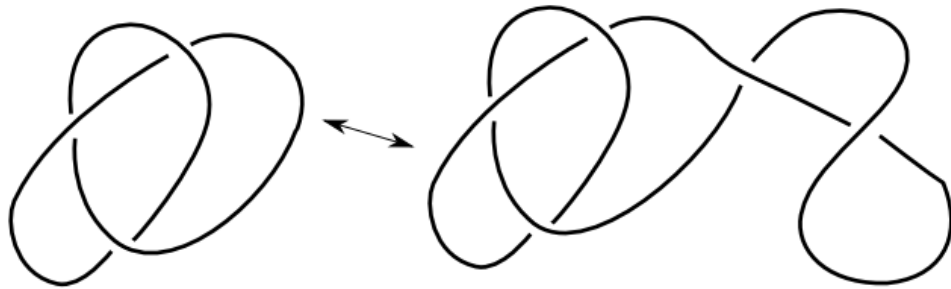
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Two knots are considered equivalent if one can be continuously deformed to coincide with the other.



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We will deal with knot projections.

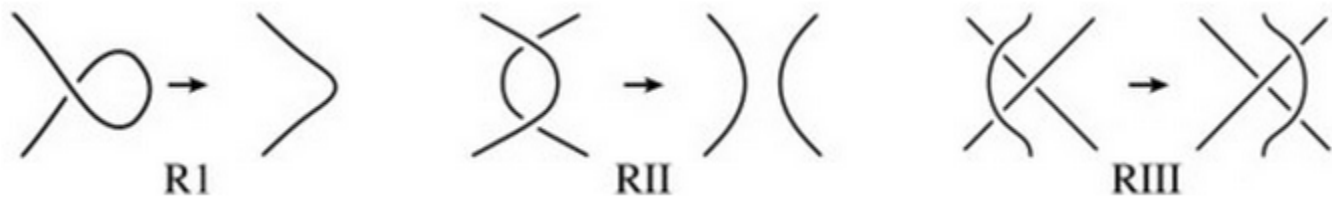


## Reidemeister's Theorem

A Reidemeister move is a local moves on a knot diagram. There are three Reidemeister moves (without considering thier variations).

# Reidemeister's Theorem

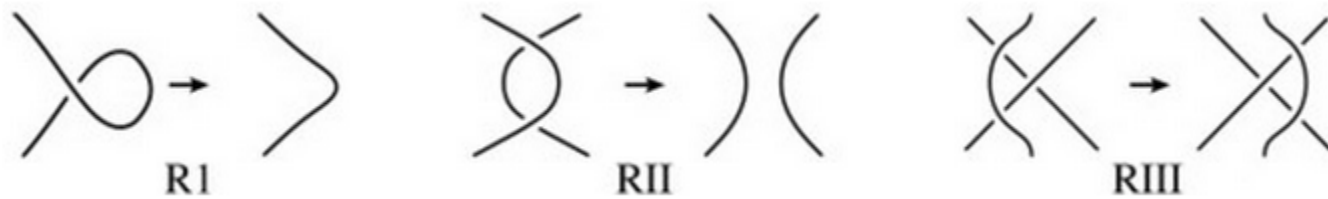
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# Reidemeister's Theorem

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**Theorem** *Two knots are equivalent iff we can get from one knot diagram to the other by a finite sequence of Reidemeister moves.*

# Knot Invariants

Denote the set of equivalence classes of links by  $\mathcal{L}$ .

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A knot invariant is a function from  $\mathcal{L}$  to some arbitrary set.

This set could be a set of number, polynomials, or even groups.

# Polynomial Knot Invariants

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The study of the Jones polynomial is a central subject in knot theory.

# The Jones Polynomial

**Definition**     *The Kauffman bracket polynomial is a function from unoriented link diagrams in the oriented plane to Laurent polynomials with integer coefficients in an indeterminate  $A$ . It maps a diagram  $D$  of a link  $L$  to  $\langle D \rangle \in \mathbb{Z}[A, A^{-1}]$  and it is characterized by the three rules:*

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$$(i) \left\langle \begin{array}{c} \diagup \quad \diagdown \\ \diagdown \quad \diagup \end{array} \right\rangle = A \left\langle \begin{array}{c} \diagup \\ \diagup \end{array} \right\rangle \left\langle \begin{array}{c} \diagdown \\ \diagdown \end{array} \right\rangle + A^{-1} \left\langle \begin{array}{c} \diagdown \\ \diagup \end{array} \right\rangle \left\langle \begin{array}{c} \diagup \\ \diagdown \end{array} \right\rangle$$

$$(ii) \left\langle D \cup \bigcirc \right\rangle = (-A^2 - A^{-2}) \langle D \rangle$$

$$(iii) \left\langle \bigcirc \right\rangle = 1$$



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$$\begin{aligned} (i) \quad \left\langle \begin{array}{c} \diagup \quad \diagdown \\ \diagdown \quad \diagup \end{array} \right\rangle &= A \left\langle \begin{array}{c} \diagup \\ \diagup \end{array} \right\rangle \left\langle \begin{array}{c} \diagdown \\ \diagdown \end{array} \right\rangle + A^{-1} \left\langle \begin{array}{c} \diagdown \quad \diagup \\ \diagup \quad \diagdown \end{array} \right\rangle \\ (ii) \quad \left\langle D \cup \bigcirc \right\rangle &= (-A^2 - A^{-2}) \langle D \rangle \\ (iii) \quad \left\langle \bigcirc \right\rangle &= 1 \end{aligned}$$

The bracket polynomial is "almost" a knot invariant. Up to change of variable and multiplication by some power of  $A$  the bracket polynomial is the Jones polynomial.

## Example

$$\langle \mathcal{G} \rangle = A \langle \mathcal{G} \rangle + A^{-1} \langle \mathcal{G} \rangle$$

## Example

$$\begin{aligned}
 \langle \text{Figure 1} \rangle &= A \langle \text{Figure 2} \rangle + A^{-1} \langle \text{Figure 3} \rangle \\
 &= A \left[ A \langle \text{Figure 2} \rangle + A^{-1} \langle \text{Figure 3} \rangle \right] + A^{-1} \left[ A \langle \text{Figure 3} \rangle + A^{-1} \langle \text{Figure 4} \rangle \right] \\
 &= A^2 \langle \text{Figure 2} \rangle + \langle \text{Figure 3} \rangle + \langle \text{Figure 3} \rangle + A^{-2} \langle \text{Figure 4} \rangle \\
 &= A^2 \left[ A \langle \text{Figure 5} \rangle + A^{-1} \langle \text{Figure 6} \rangle \right] + A \langle \text{Figure 5} \rangle + A^{-1} \langle \text{Figure 6} \rangle \\
 &\quad + A \langle \text{Figure 7} \rangle + A^{-1} \langle \text{Figure 8} \rangle + A^{-2} \left\{ A \langle \text{Figure 8} \rangle + A^{-1} \langle \text{Figure 9} \rangle \right\} \\
 &= A^3 \langle \text{Figure 5} \rangle + A \langle \text{Figure 6} \rangle + A \langle \text{Figure 6} \rangle + A^{-1} \langle \text{Figure 6} \rangle \\
 &\quad + A \langle \text{Figure 7} \rangle + A^{-1} \langle \text{Figure 8} \rangle + A^{-1} \langle \text{Figure 8} \rangle + A^{-3} \langle \text{Figure 9} \rangle \\
 &= A^3(-A^2 - A^{-2})^2 + A(-A^2 - A^{-2}) + A(-A^2 - A^{-2}) \\
 &\quad + A^{-1} + A(-A^2 - A^{-2}) + A^{-1} + A^{-1} + A^{-3}(-A^2 - A^{-2}) \\
 &= A^7 - A^3 - A^{-5}
 \end{aligned}$$

## The Temperley-Lieb Algebra

Consider the disk  $D^2$  as a rectangle with 4 designated points on the top and 4 designated points on the bottom.

## The Temperley-Lieb Algebra

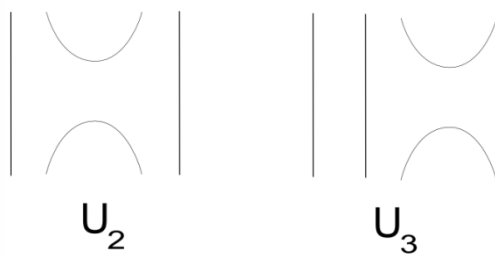
Consider the disk  $D^2$  as a rectangle with 4 designated points on the top and 4 designated points on the bottom.

In  $(D^2, 8)$  define the elementary tangles  $1_n, U_1, U_2, U_3$  and  $U_4$  by

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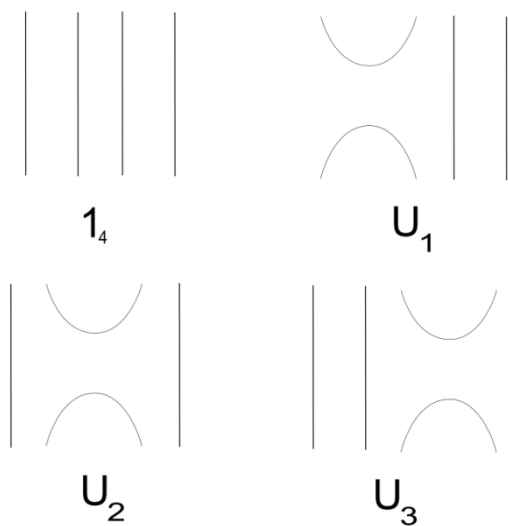
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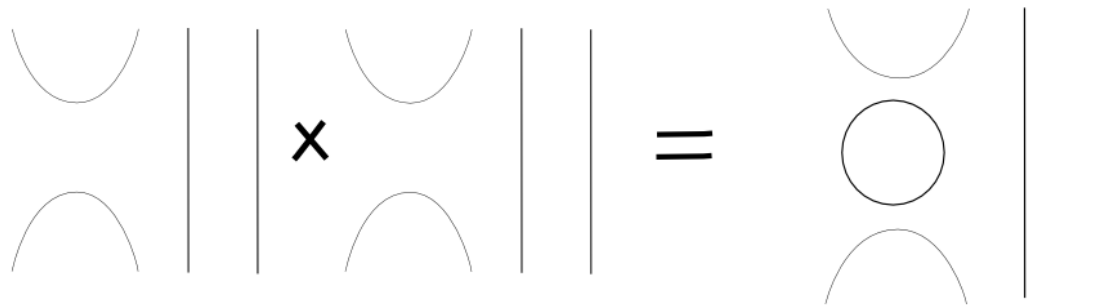


Similarly, consider the disk  $D^2$  as a rectangle with  $n$  designated points on the top and  $n$  designated points on the bottom.

In  $(D^2, 2n)$  define the elementary tangles  $1_n, U_1, \dots, U_{n-1}$ .

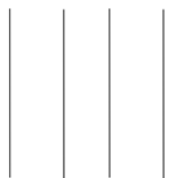
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We define "multiplication" in  $(D^2, 2n)$  as follows :





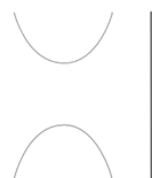
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$1_4$



$U_1$

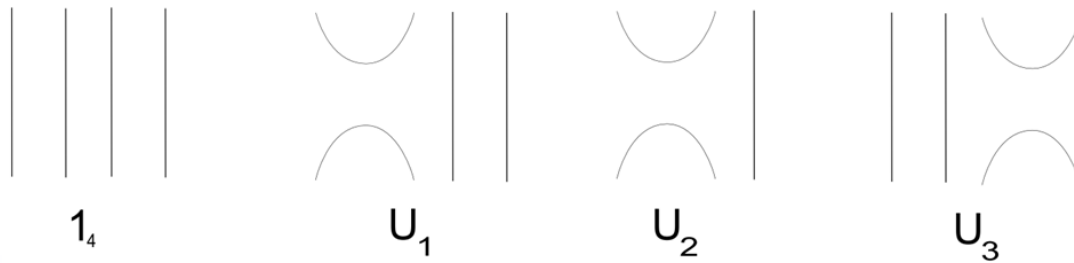


$U_2$



$U_3$

# The Temperley-Lieb Algebra

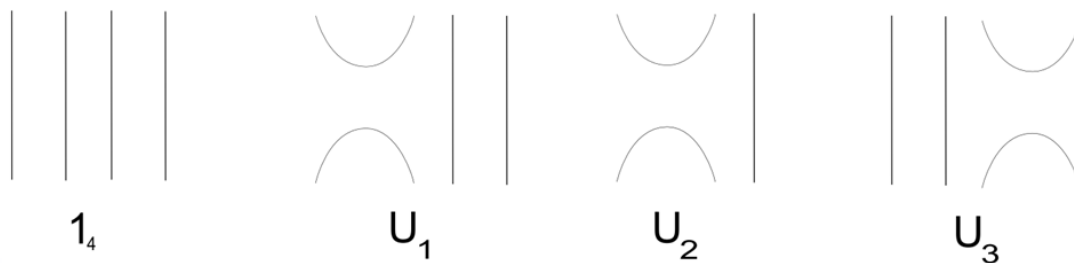


There are some relations between the  $U_i$ 's :

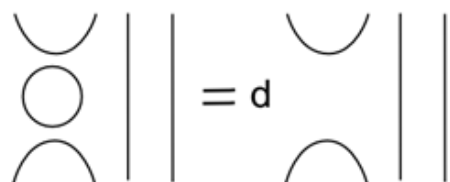
Diagrammatic equation:  $U_i U_i = d U_i$

The diagram shows two vertical parallel lines with two arcs connecting them, one above and one below, forming a circle. This is equal to  $d$  times the diagram of two vertical parallel lines with two arcs connecting them, one above and one below.

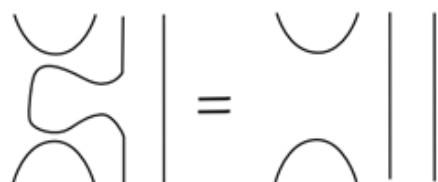
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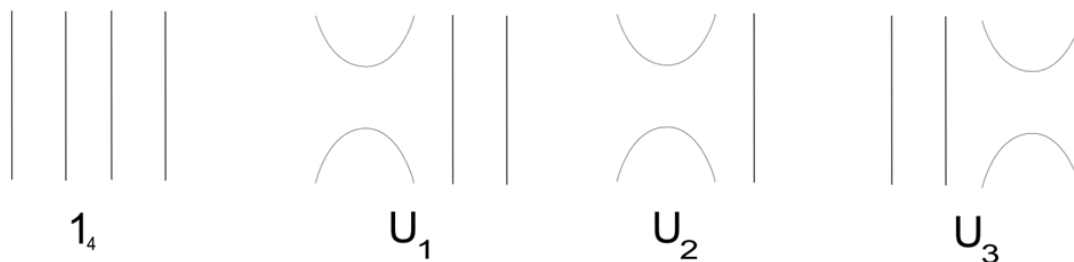


$$U_i U_i = d U_i$$

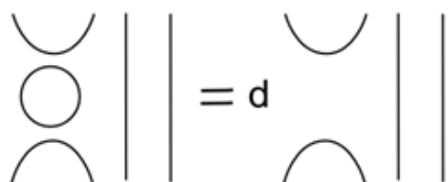


$$U_i U_{i\pm 1} U_i = U_i$$

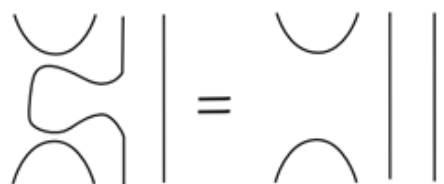
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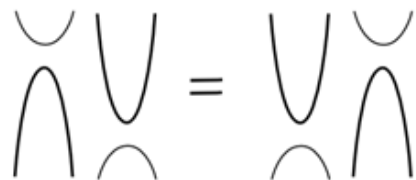
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$$U_i U_i = d U_i$$



$$U_i U_{i\pm 1} U_i = U_i$$



$$U_i U_j = U_j U_i, |i - j| > 1$$

Where  $d = -A^2 - A^{-2}$

# The Jones-Wenzel Idempotent

In the Temperley-Lieb algebra there exists an element of fundamental importance called the Jones-Wenzel Idempotent and this element is defined diagrammatically as follows

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$$\begin{array}{c} \square \\ | \\ n+1 \end{array} = \begin{array}{c} \square \\ | \\ n \end{array} + \left( \frac{\Delta_{n-1}}{\Delta_n} \right) \begin{array}{c} \square \\ | \\ n-1 \\ \square \\ | \\ n \end{array} \begin{array}{c} | \\ 1 \\ \square \\ | \\ 1 \end{array}, \quad \begin{array}{c} \square \\ | \\ 1 \end{array} = \begin{array}{c} | \\ 1 \end{array}$$

$$\Delta_n := (-1)^n \frac{A^{2(n+1)} - A^{-2(n+1)}}{A^2 - A^{-2}}$$

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$$\square_{n+1} = \square_n \mid 1 - \left( \frac{\Delta_{n-1}}{\Delta_n} \right) \left( \begin{array}{c} n \\ \text{---} \\ n-1 \end{array} \cup \begin{array}{c} n-1 \\ \text{---} \\ n \end{array} \right), \quad \square_1 = \mid_1$$

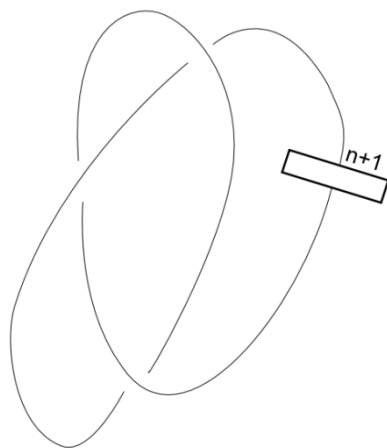
$$\Delta_n := (-1)^n \frac{A^{2(n+1)} - A^{-2(n+1)}}{A^2 - A^{-2}}$$

## Example

$$\square_2 = \mid \mid - \frac{1}{d} \left( \begin{array}{c} 1 \\ \text{---} \\ 1 \end{array} \right), \quad d = -A^2 - A^{-2}$$

# The Colored Jones Polynomial

Up to change of variable and multiplication by some power of  $A$  the  $n$ th colored Jones polynomial equals for the a knot  $K$  equals to:





# The Volume Conjecture

One reason why the study the colored Jones polynomial is interesting is the volume conjecture. The volume conjecture states :

$$\lim_{N \rightarrow \infty} \left( \frac{2\pi \log | \langle K \rangle_N |}{N} \right) = \text{vol}(K)$$

where  $\text{vol}(K)$  denotes the hyperbolic volume of the complement of  $K$  in the 3-sphere.

**Thanks**