An Introduction to The Colored Jones Polynomial

LSU Math Geaux Seminar

Mustafa Hajij

Basic Definitions

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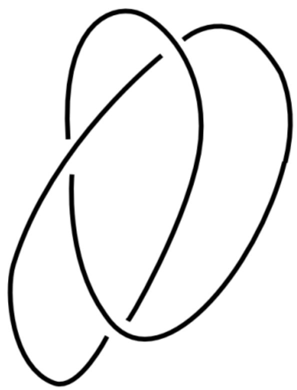

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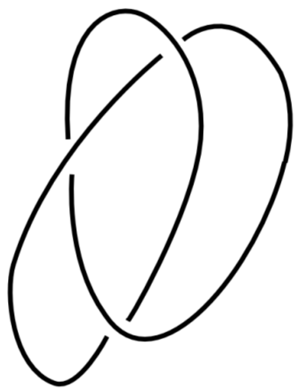
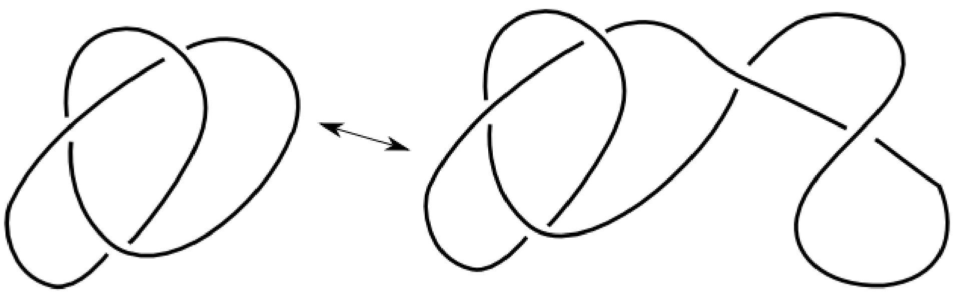


A link is a smooth embedding of disjoint union of finite number of S^1 in $\mathbb{R}^3$.

Basic Definitions

Two knots are considered equivalent if one can be continuously deformed to coincide with the other.



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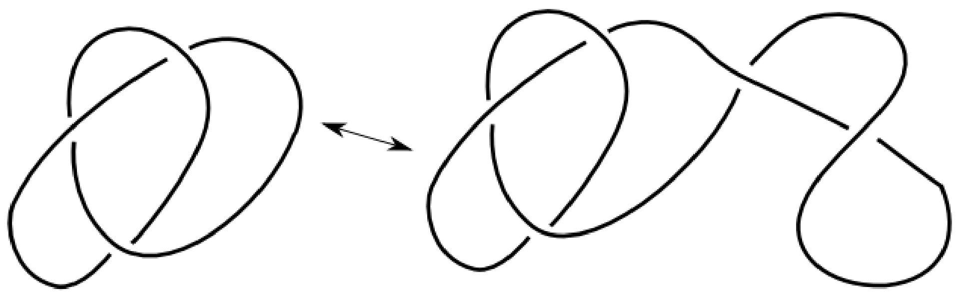

We will deal with knot projections.

Reidemeister's Theorem

A Reidemeister move is a local moves on a knot diagram. There are three Reidemeister moves (without considering thier variations).

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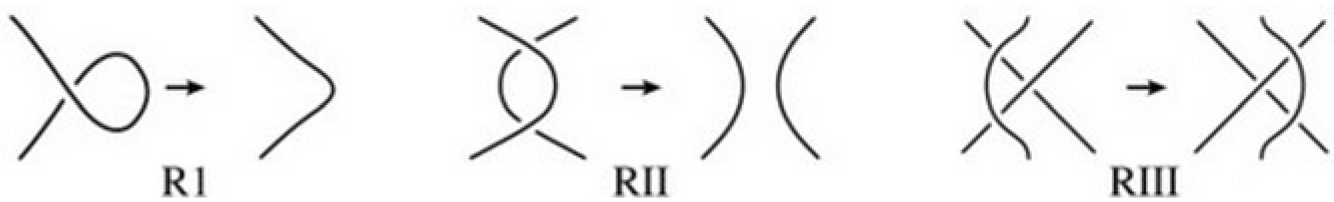

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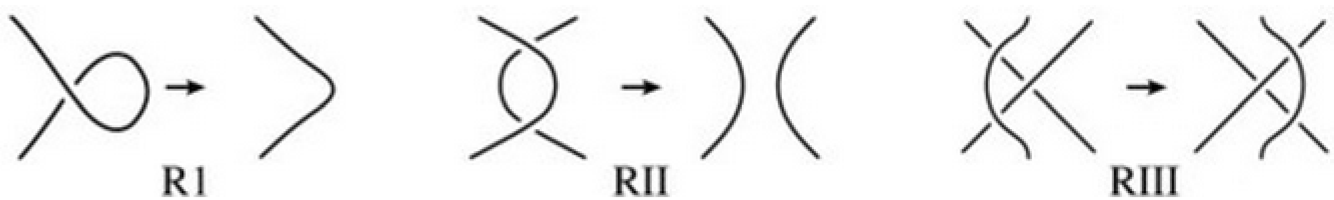

Theorem *Two knots are equivalent iff we can get from one knot diagram to the other by a finite sequence of Reidemeister moves.*

Knot Invariants

Denote the set of equivalence classes of links by $\mathcal{L}$.

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This set could be a set of number, polynomials, or even groups.

Polynomial Knot Invariants

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The study of the Jones polynomial is a central subject in knot theory.

The Jones Polynomial

Definition *The Kauffman bracket polynomial is a function from unoriented link diagrams in the oriented plane to Laurent polynomials with integer coefficients in an indeterminate* A. *It maps a diagram* D *of a link* L *to* $\langle D \rangle \in \mathbb{Z}[A, A^{-1}]$ *and it is characterized by the three rules:*

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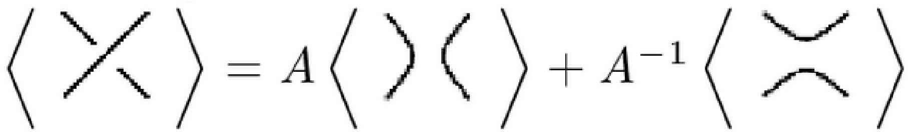
(i) 

(ii) $$\left\langle D \cup \bigcirc \right\rangle = (-A^2 - A^{-2}) \langle D \rangle$$

(iii) $$\left\langle \bigcirc \right\rangle = 1$$

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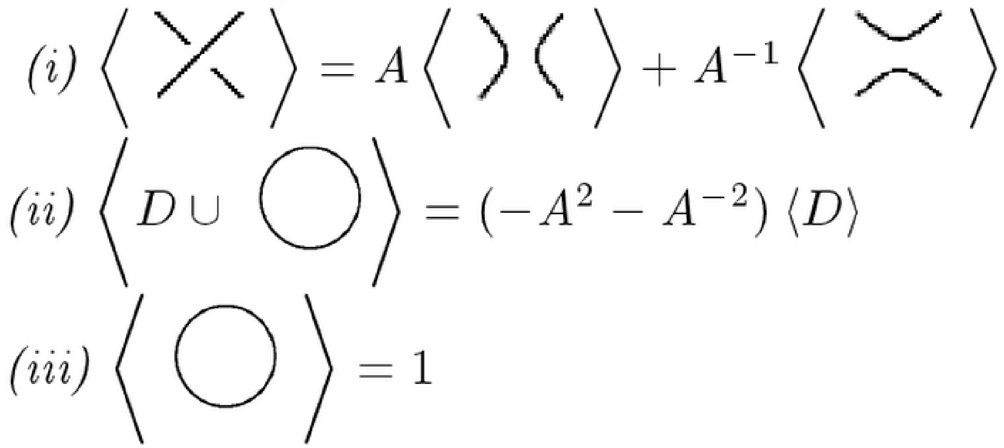

The bracket polynomial is "almost" a knot invariant. Up to change of variable and multiplication by some power of A the bracket polynomial is the Jones polynomial.

Example

$\langle$  $\rangle = A \langle$  $\rangle + A^{-1} \langle$  $\rangle$

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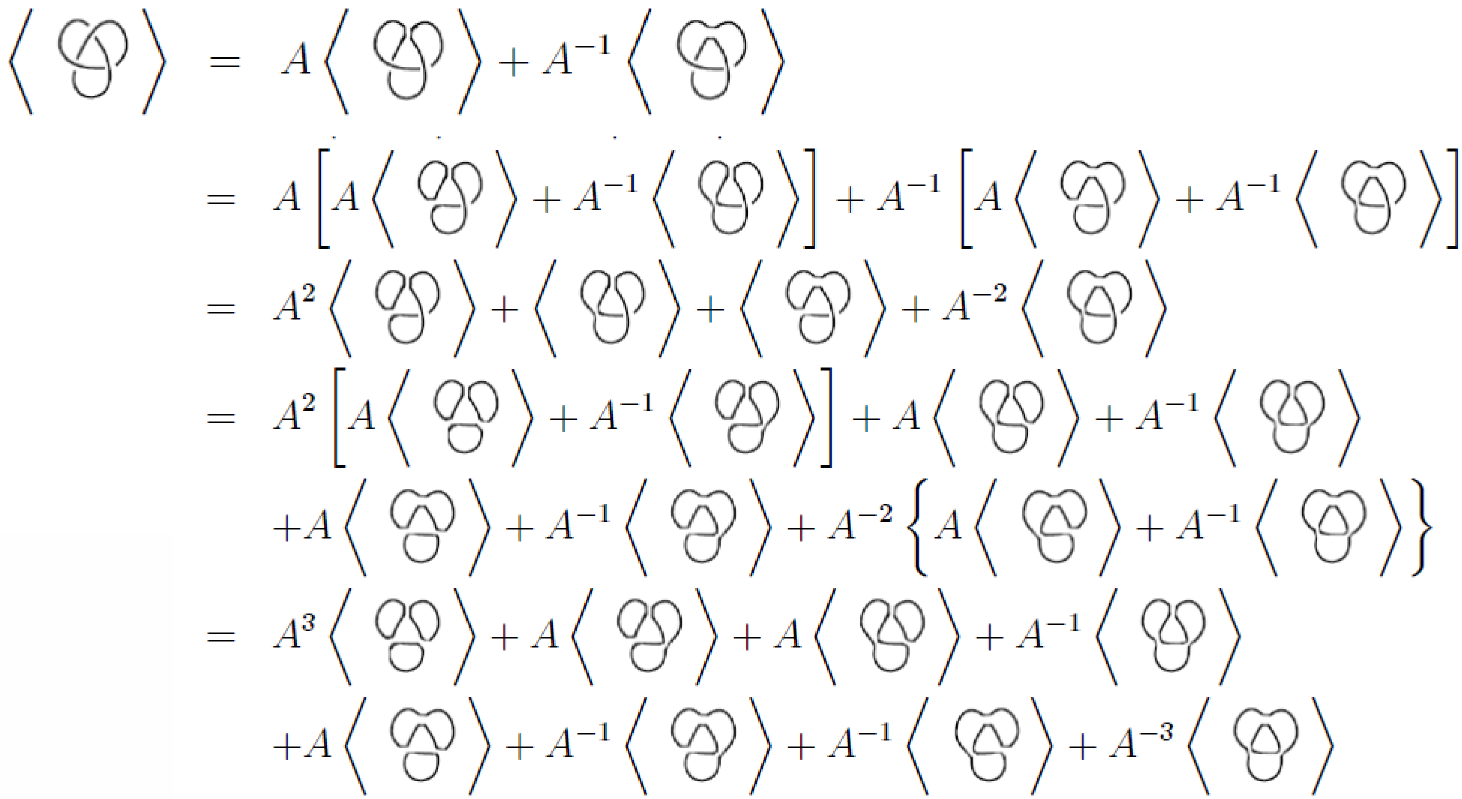

$$= A^3(-A^2 - A^{-2})^2 + A(-A^2 - A^{-2}) + A(-A^2 - A^{-2})$$

$$+A^{-1} + A(-A^2 - A^{-2}) + A^{-1} + A^{-1} + A^{-3}(-A^2 - A^{-2})$$

$$= A^7 - A^3 - A^{-5}$$

The Temperly-Lieb Algebra

Consider the disk D^2 as a rectangle with 4 designated points on the top and 4 designated points on the bottom.

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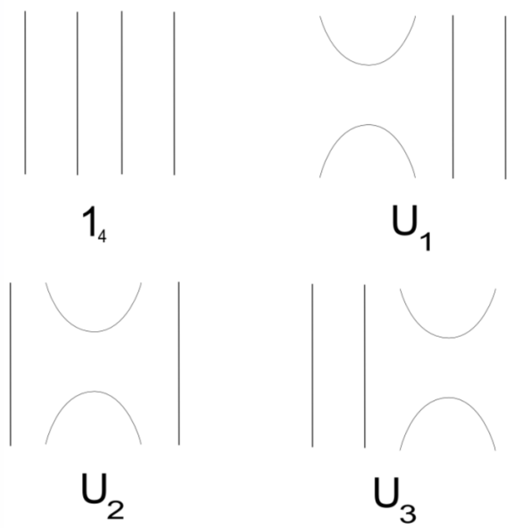

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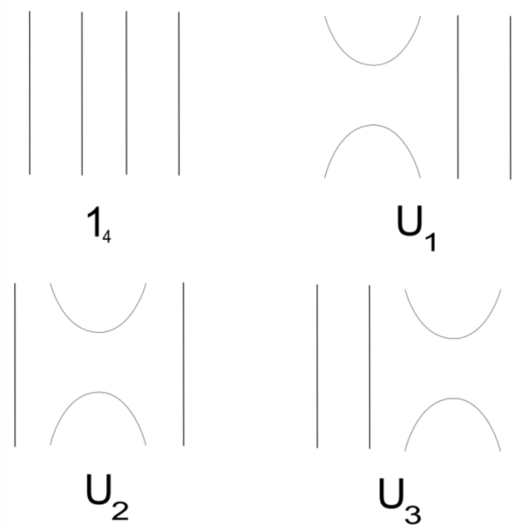
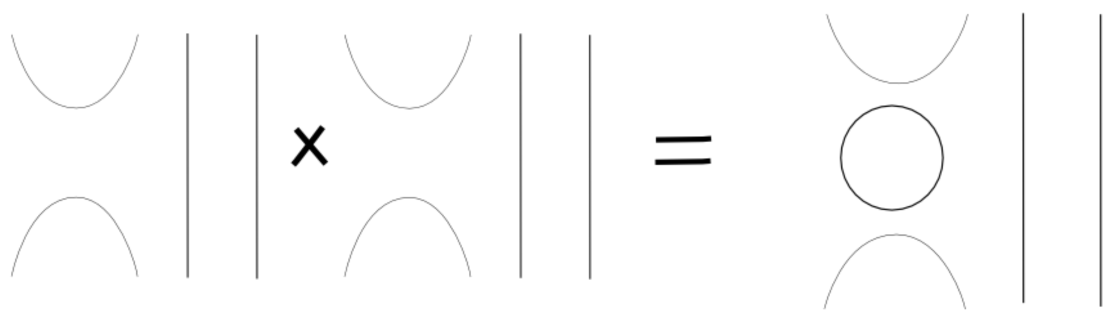


Similarly, consider the disk D^2 as a rectangle with n designated points on the top and n designated points on the bottom.

In $(D^2, 2n)$ define the elementary tangles $1_n, U_1, ..., U_{n-1}$.

The Temperly-Lieb Algebra

We define "multiplication" in $(D^2, 2n)$ as follows :



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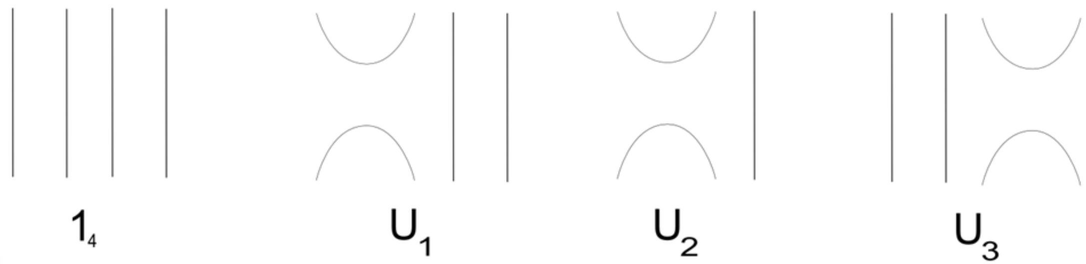
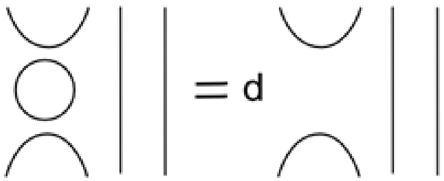
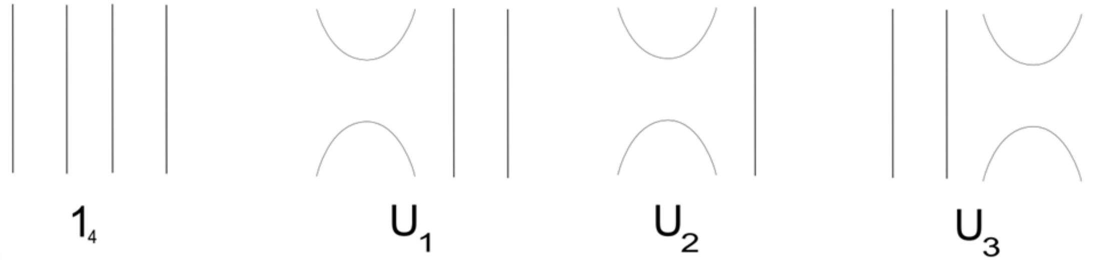


There are some relations between the U_i's :



$$U_i U_i = d U_i$$

The Temperly-Lieb Algebra



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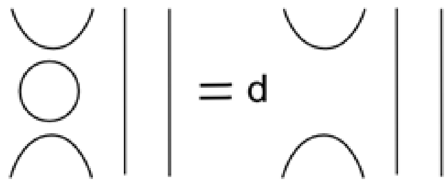

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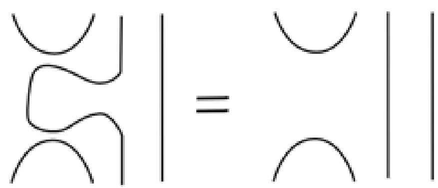

$$U_i U_{i\pm 1} U_i = U_i$$

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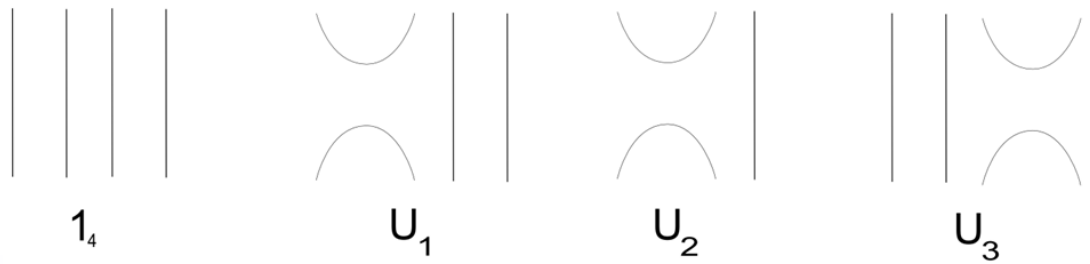

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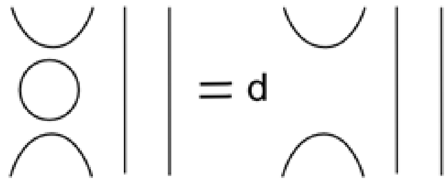

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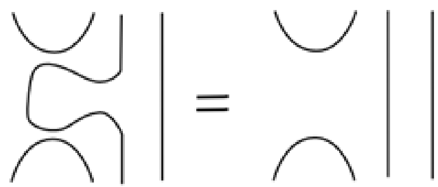

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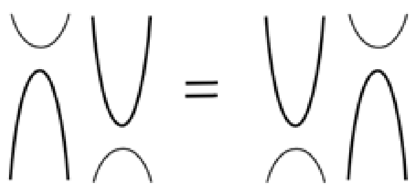

$$U_i U_j = U_j U_i,\ |i-j| > 1$$

Where $d = -A^2 - A^{-2}$

The Jones-Wenzel Idempotent

In the Temperley-Lieb algebra there exists an element of fundamental importance called the Jones-Wenzel Idempotent and this element is defined diagrammatically as follows

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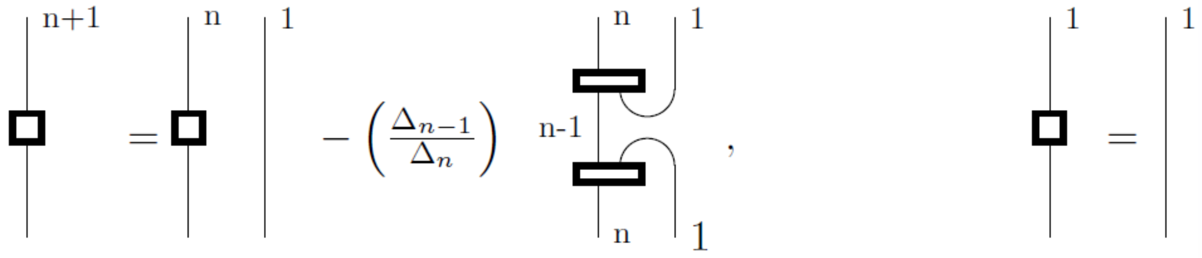

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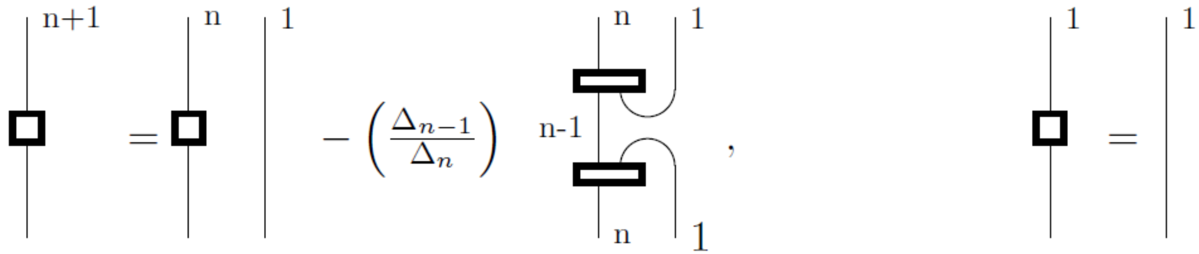

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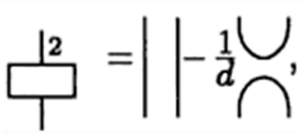
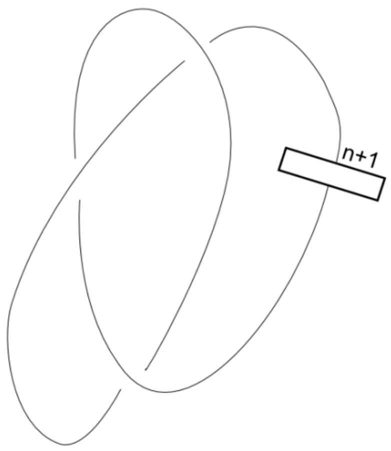


$d = -A^2 - A^{-2}$

The Colored Jones Polynomial

Up to change of variable and multiplication by some power of A the nth colored Jones polynomial equals for the a knot K equals to:



The Volume Conjecture

One reason why the study the colored Jones polynomial is interesting is the volume conjecture. The volume conjecture states :

$$lim_{N\to\infty}(\frac{2\pi\log|<K>_N|}{N}) = vol(K)$$

where $vol(K)$ denotes the hyperbolic volume of the complement of K in the 3-sphere.

Thanks