Mathematical Billiards

Junior Topology Siminar

Mustafa Hajij

What is a mathematical billiard?

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·The angel of incidence =The angle of reflection

We assume that the reflection occurs at a smooth point of the boundary.

If the billiard ball hits a "corner" then the reflection is not well-defined and the motion of the ball terminates right there.

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Definition *A billiard trajectory in a smooth body domain* $D \subset \mathbb{R}^n$ *is a polygon* $P \subset D$ *with its vertices on the boundary of* D*, and at each vertex the direction of line changes according to the elastic reflection rule.*

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Definition *A billiard trajectory* P *is called* n*-periodic if the number of vertices of* P *is* n*.*

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c) Let D be a convex domain with smooth boundary in three-dimensional space. Find at least three distinct 2-periodic billiard orbits in D.

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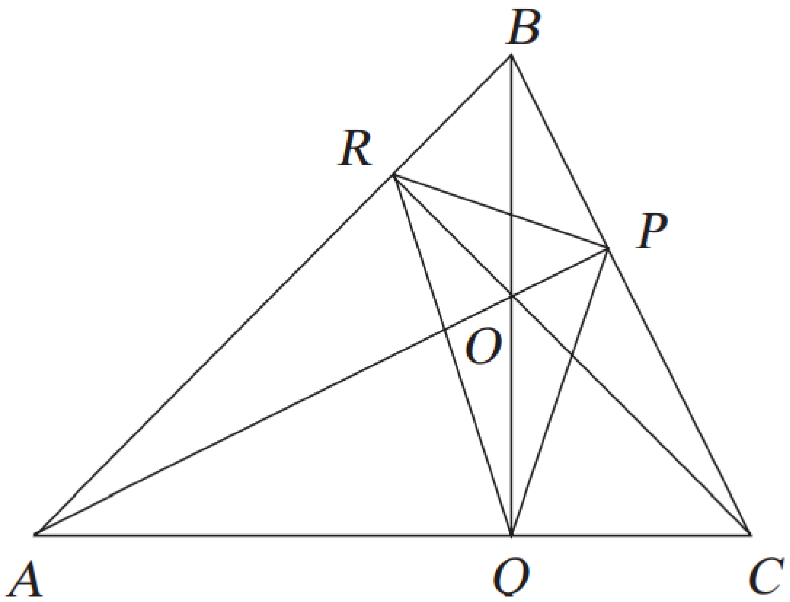
Some times billiard trajectories show up in "families" : A disc D in the plane contains a one parameter family of 2-periodic billiard trajectories. Namely the diameters of D.

Example

Lemma *The triangle connecting the base points of the three altitudes is a 3-periodic billiard trajectory.*



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The space of inscribed n-gons in ∂D can be characterized by the following definition

Let X be a topological space and n be a positive integer. Denote by $G(X, n)$ the subspace of the Cartesian power $X^n = X \times X \times \cdots \times X$ consisting of all configurations $(x_1, x_2, ..., x_n)$ such that $x_i = x_{i+1}$ for $i = 1, 2, ..., n-1$ and $x_n = x_1$. The space $G(X, n)$ is called the cyclic configuration space.

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Recall that a disc D in the plane contains a family of 2-periodic billiard trajectories : the diameters of D. It seems that we should consider all this family to be just one single billiard and so we give the following definition.

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The dihedral group D_n acts on the cyclic configuration space; its action is generated by the cyclic permutation and the reflection

$$(x_1, x_2, \ldots, x_n) \mapsto (x_2, x_3, \ldots, x_n, x_1), \quad (x_1, x_2, \ldots, x_n) \mapsto (x_n, x_{n-1}, \ldots, x_1).$$

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Under this action our question can be reformulated as follows: Count how many (or find a lower bound for the) distinct D_n-orbits in the cyclic configuration space $G(X, n)$.

The Perimeter Length Function

Definition *Let $X \subset R^{m+1}$ be a smooth hypersurface. Then X is n-generic if $L_X : G(X,n) \to R$ is a Morse function.*

Let $X \subset R^{m+1}$ be a smooth closed strictly convex hypersurface, topologically the sphere, which is the boundary of the billiard table. Denote by

$$L_X : G(X,n) \to \mathbb{R}$$

the perimeter length function defined by,

$$L_X(x_1, x_2, ..., x_n) = -|x_1 - x_2| - |x_2 - x_3| - ... - |x_n - x_1|,$$

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Theorem *The n-periodic billiard orbits in X are precisely the critical points of the function L_X*

Identifying $G(X, n)$ with $G(S^m, n)$, we see that the shape of the billiard domain X becomes encoded in the function $L_X : G(S^m, n) \to \mathbb{R}$, and, from the previous theorem, the problem of finding the closed billiard trajectories inside X turns into a Morse theory problem.

Some Definitions

Definition *Let* $(V, ||.||)$ *be a normed topological vector space. A strictly convex space is one for which, given any two points* x *and* y *in the boundary* ∂B *of the unit ball* B *of* V*, the affine line* $L(x, y)$ *passing through* x *and* y *meets* ∂B *only at* x *and* y*.*

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A lower bound for the number of trajectories

Theorem *Let $X \subset R^{m+1}$ be a smooth strictly convex hypersurface, where $m \geq 3$. Fix an odd number $n \geq 3$. Then*

(A) The number of distinct D_n-orbits of n-periodic billiard trajectories inside X is not less than $[log_2(n-1)] + m$.

(B) For a generic X, the number of distinct D_n-orbits of n-periodic billiard trajectories inside X is not less than $(n-1)m$

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(3) Replacing the replacing $G(X, n)$ by a "better space". Namely a compact manifold with boundary $G_\epsilon(X, n) \subset G(X, n)$. The space$G_\epsilon(X, n)$ is D_n-equivariant homotopy equivalence and it contains all the critical points of L_X.

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(4) Using cup-length of $H^*(G(S^m, n); \mathbb{Z}_2)$ and Morse theory methods to obtain a lower bound on the closed billiard trajectories.

Other Lower bounds

Theorem *(Farber, M) Let* $n \geq 3$ *be an odd prime and* X *be a smooth strictly convex closed surface in* 3*-space. Then the number of distinct* D_n*-orbits of* n*-periodic billiard trajectories inside* X *is not less than* $(n+1)/2$*.*

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Theorem *(Michael Farber ·· Serge Tabachnikov) Let* $n \geq 3$ *be an odd number and let* X *be a generic smooth strictly convex closed surface in 3-space. Then the number of distinct* D_n *orbits of* n*-periodic billiard trajectories inside* X *is not less than* $2(n-1)$.

Ref.

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[3]Tabachnikov, Serge, Geometry and Billiards, AMS, 2005.

Thank You