Orientation on Manifolds

Mustafa Hajij

The Notion of Orientation

- The notion of orientation appears in many different contexts.

In everyday conversation.

In Physics.

- In Mathemtics

Vector spaces.

Differentiable manifolds.

Topological manifolds.

The Notion of Orientation

- How would we define an orientation on $\mathbb{R}^2$?

Criteria that we want to have in the definition:

1. Orientation should be *preserved* under *rotation*.
2. Orientation should be *reserved* under *reflection*.

- Note that the notions of "clockwise" and "counterclockwise" have this property.

The Notion of Orientation

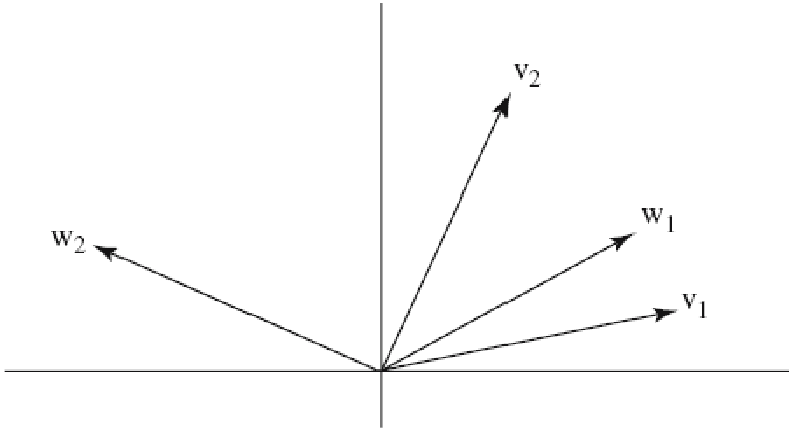
Say we have (w1,w2) an ordered basis of $\mathbb{R}^2$.

We could use this ordered basis to talk about CCW motion in $\mathbb{R}^2$.

The pairs (v1,v2) in

Also corresponds

to CCW motion.



The Notion of Orientation

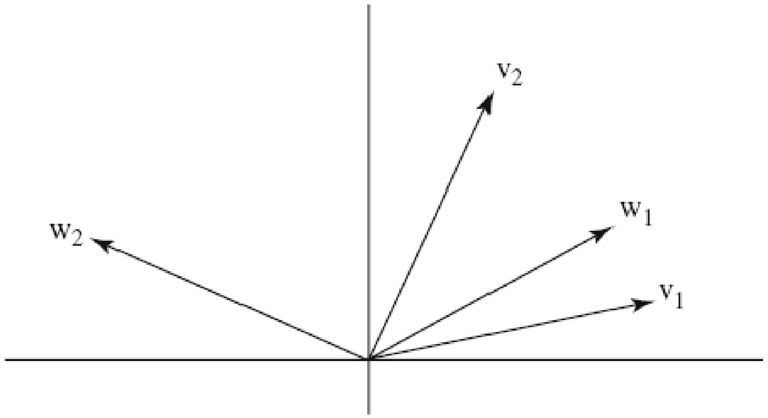
- This suggests that we need to define some appropriate equivalence relation.

$$v_i = a_{i1} w_1 + a_{i2} w_2$$

$$\downarrow$$

$$(a_{ij})$$

- Define (w1,w2) to be equivalent to (v1,v2) if the determinant of the matrix (aij) is positive.



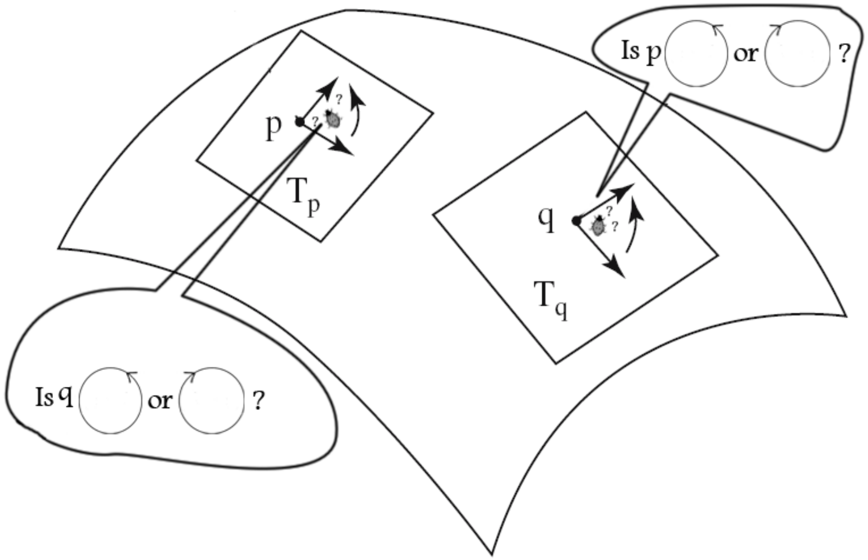
The Notion of Orientation

An orientable surface is one where it is possible to define a *consistent* notion of left and right or clockwise and counterclockwise.

But what does “consistent” mean?

The Notion of Orientation

If two persons are standing at different points of a surface and they each have decided what to call clockwise, how can they determine whether their choices are consistent (assuming that they cannot see each other)?



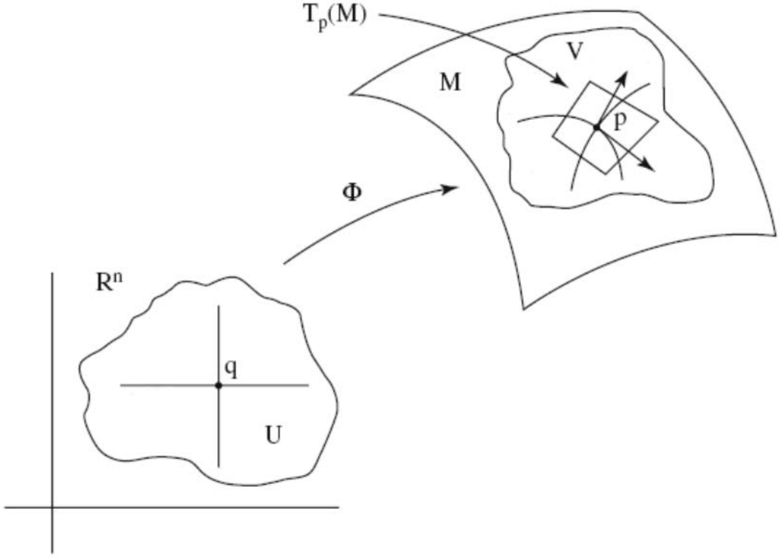
The Notion of Orientation

One way to answer this question is to have one of them *walk over* to where the other one is standing and then *compare* their notions of clockwise.

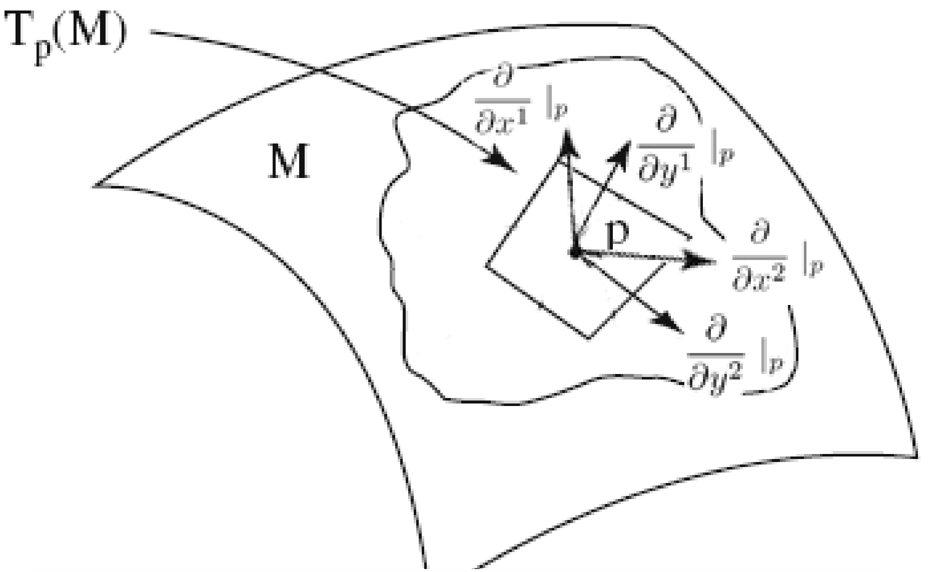
Orientation on Differntiable Manifolds

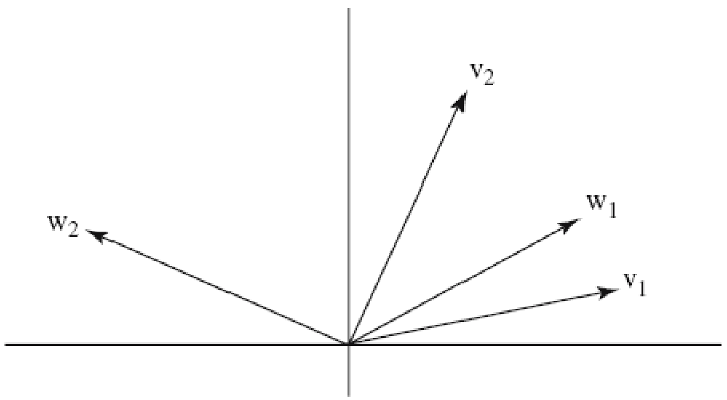
Let M be an $n-$dimensional manifold, and let (U, ϕ) be a chart containing p in M, then we know that the tangent space T_pM has basis

$$\frac{\partial}{\partial x^1}\ |_p, \frac{\partial}{\partial x^2}\ |_p, .., \frac{\partial}{\partial x^n}\ |_p,$$



Orientation on Differntiable Manifolds





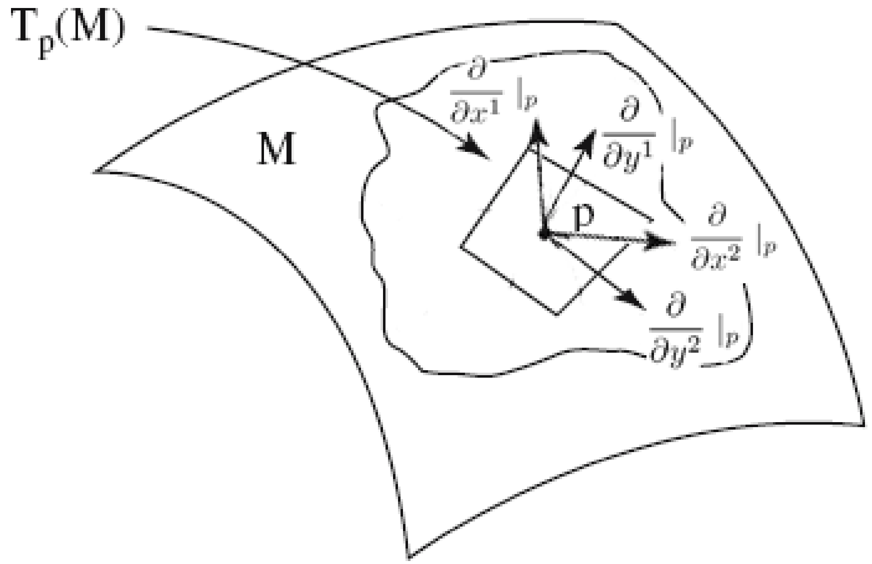
$$\frac{\partial}{\partial y^j}\Big|_p = \frac{\partial x^1}{\partial y^j}\Big|_p \frac{\partial}{\partial x^1}\Big|_p + \frac{\partial x^2}{\partial y^j}\Big|_p \frac{\partial}{\partial x^2}\Big|_p$$

$$\left(\frac{\partial x^i}{\partial y^j}\Big|_p\right)$$

$$v_i = a_{i1} w_1 + a_{i2} w_2$$

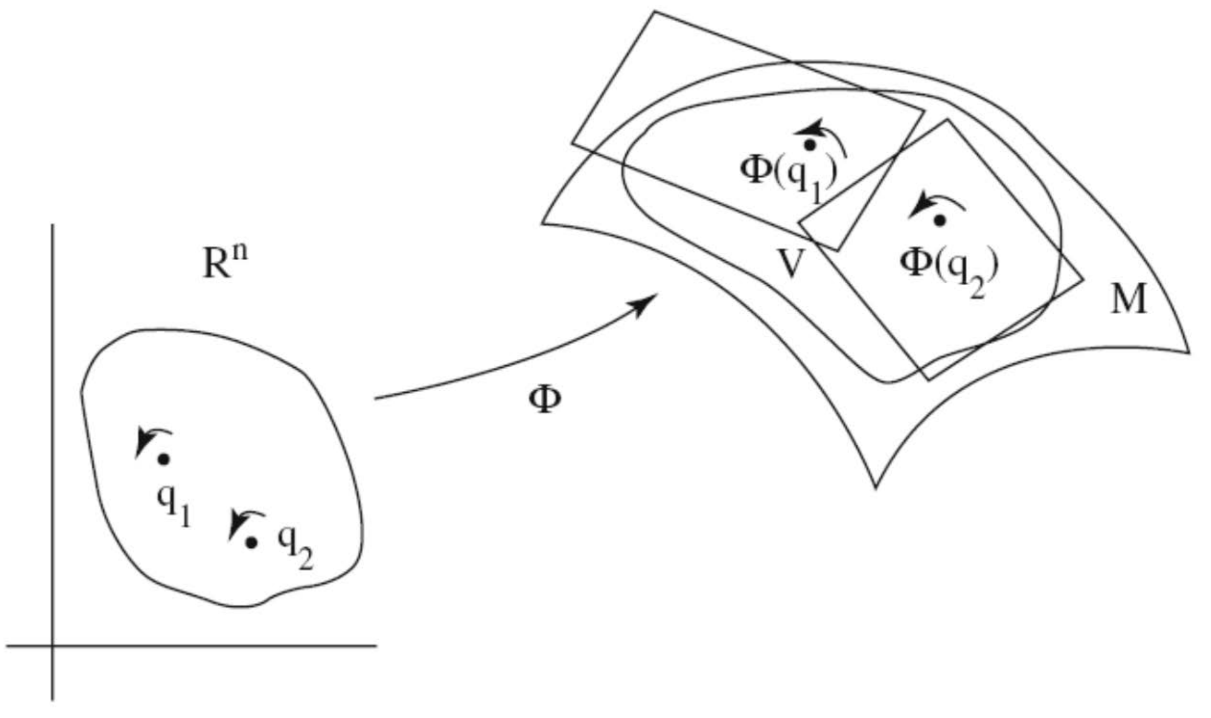
$$(a_{ij})$$

Orientation on Differntiable Manifolds



Define $(\frac{\partial}{\partial x^1}|_p, \frac{\partial}{\partial x^2}|_p, .., \frac{\partial}{\partial x^n}|_p)$ to be equivalent to $(\frac{\partial}{\partial y^1}|_p, \frac{\partial}{\partial y^2}|_p, .., \frac{\partial}{\partial y^n}|_p)$ if the determinant of the matrix $det(\frac{\partial x^i}{\partial y^j}|_p) > 0$ is positive.

Orientation on Differntiable Manifolds



Continuously varying choice of orientation.

Orientation on Differntiable Manifolds

A definition of orientation on a differentiable Manifold

There is a collection $\Phi = \{(V,\psi)\}$ *of coordinate systems on* M *such that*

$$(1) \qquad M = \bigcup_{(V,\psi)\in\Phi} V \qquad \text{and} \qquad \det\left(\frac{\partial x_i}{\partial y_j}\right) > 0 \quad \text{on} \quad U \cap V$$

whenever $(U, x_1, \ldots, x_n)$ *and* $(V, y_1, \ldots, y_n)$ *belong to* Φ.

Orientation on Differntiable Manifolds

Other ways to define orientation on a differntiable manifold:

1. Using *differential forms*, useful in intergration on manifolds.
2. Using the *exterior n-bundle* of the manifold M, useful when we do not want to deal with coordinates.

Orientation on Differntiable Manifolds

Theorem *A manifold* M *of dimensional* n *is orientable iff it has a* C^∞ *nowhere vanishing* $n-$*form.*

Example *(a) Every Lie group* G *is orientable, for* $\omega_1, ..., \omega_n$ *is a basis for left invariant* $1-$*form on* G*, then* $\omega_1 \wedge ... \wedge \omega_n$ *is a global nowhere vanishing* $n-$*form on* G*.*

(b) The standard orientation on the Euclidean space $\mathbb{R}^n$ *is the one determined by the* $n-$*form* $dx_1 \wedge ... \wedge dx_n$*.*

Orientation on Topological Manifolds

- Can we define an orientation on a toplogical manifold? In other words, can we define an orienatation on a manifold using only its topological structure?

Orientation on Topological Manifolds

Consider the local homology at the origin of $\mathbb{R}^2$

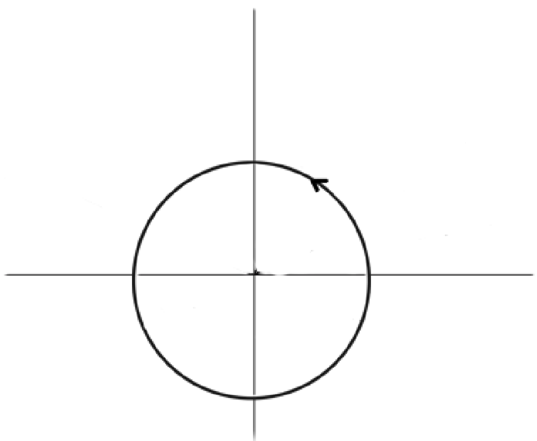
$$\begin{aligned} H_2(\mathbb{R}^2, \mathbb{R}^2 - \{0\}, \mathbb{Z}) &= \hat{H}_1(\mathbb{R}^2 - \{0\}, \mathbb{Z}) \\ &= \hat{H}_1(S^1, \mathbb{Z}) \end{aligned}$$

Note that a choice of a generator of the infinite cyclic group $H_2(\mathbb{R}^2, \mathbb{R}^2 - \{0\}, \mathbb{Z})$ corresponds to a choice of a generator of the infinite cyclic group $\hat{H}_1(S^1, \mathbb{Z})$ which in turn has the notion of CW and CCW in $\mathbb{R}^2$.

Orientation on Topological Manifolds

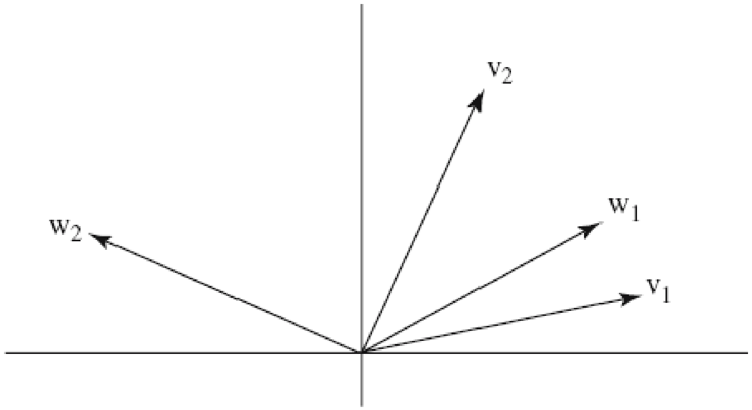
Definition *An orientation of an* $n-$*dimensional topologocal manifold* M *is a function* $f\!:\!M \to \Sigma$ *assigning to each* $x \in M$ *a local orientation* $\mu_x \in H_n(M; M - \{x\})$, *satisfying the 'local consistency' condition that each* $x \in M$ *has a neighborhood* $\mathbb{R}^n \subset M$, *containing an open ball* B *of finite radius about* x *such that the following condition is satisfied : if* $y \in B$, *then* $f(y) = \mu_y = \mu_B$, *where* μ_B *is the generator of* $H_n(M, M - B) = H_n(\mathbb{R}^n, \mathbb{R}^n - B)$. *If an orientation exists for* M, *then* M *is called orientable.*

Compatibility Between Different Notions of Orientation



A choice of a generator of

$$H_2(\mathbb{R}^2, \mathbb{R}^2 - \{0\}, \mathbb{Z})$$



A choice an ordered basis of $\mathbb{R}^2$

Compatibility Between Different Notions of Orientation

- One always has to address the issue of *compatibility* between the different definitions.
- Is there one general unifying condition for an n-dimensional connected manifold M to be orientable?

Compatibility Between Different Notions of Orientation

*Theorem: A closed compact connected n-dimensional differentiable manifold **M** is orientable if and only if* $H_n(M) = \mathbb{Z}$

- Step 1: Relate an orientation of a *tangent plane* at a point to *local homology* of the point in *M*.
- Step 2: Relate a "continuously varying" collection of *local orientations* to a *homology class*.

Thank You