Classical and Quantum Representations of Braid Groups From sl2c

Junior Topology Seminar

Mustafa Hajij

U (sl_2) Modules

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$$[H,E] = 2E,$$

$$[H,F] = -2F,$$

$$[E,F] = H$$

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In other words $E(V_\alpha) \subset V_{\alpha+2}$ and $F(V_\alpha) \subset V_{\alpha-2}$.

U (sl_2) Modules

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Suppose that the eigenvalue of v is λ and consider the the vector space spanned by

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It is not hard to show that this actually a submodule of V and by irreducibility we conclude that this vector space is V itself.

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$$
\begin{aligned}
E(v) &= x\frac{\partial}{\partial y}(v) \\
F(v) &= y\frac{\partial}{\partial x}(v) \\
H(v) &= (x\frac{\partial}{\partial y} - y\frac{\partial}{\partial x})(v)
\end{aligned}
$$

$U(sl_2)$ Simple Modules

Theorem

$(1) V_n \quad is\ a\ simple\ U(sl(2,\mathbb{C}))\ module$

U (sl_2) Simple Modules

Theorem

(1)V_n *is a simple* $U(sl(2,\mathbb{C}))$ *module*

(2) *Every finite dimensional simple module* $U(sl(2,\mathbb{C}))$ *is isomorphic to one of* V_n

U (sl_2) morphisms

Definition *Given two* $U\ (sl(2))-$*modules* V *and* W*, an intertwiner from* V *to* W *is a linear map* $\phi : V \to W$ *such that*

$$\phi \circ E_V = E_W \circ \phi$$

$$\phi \circ F_V = F \circ_W \phi$$

$$\phi \circ H_V = H_W \circ \phi.$$

The set of all morphisms from V *to* W *will be denoted* $Hom_{U(sl(2))}(V, W)$.

Classical Temperley-Lieb Algebra

Let $V := V_1 = span\{x, y\}$ be the $U(sl(2))-$module defined eariler. Suppose that $| : V \to V$ is the identity morphism.

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Define $\cap : V \otimes V \to \mathbb{C}$ via

$$
\begin{aligned}
\cap(x \otimes x) &= 0 \\
\cap(x \otimes y) &= i \\
\cap(y \otimes x) &= -i \\
\cap(y \otimes y) &= 0
\end{aligned}
$$

Classical Temperley-Lieb Algebra

Define $\cup : \mathbb{C} \to V \otimes V$ by

$$\cup(1) = ix \otimes y - iy \otimes x$$

Classical Temperley-Lieb Algebra

Lemma *(1) The maps* $\cap$ *and* $\cup$ *are intertwiner operators for* $U(sl(2))$.

(2) The maps $\cap \circ \cup = \bigcirc : \mathbb{C} \to \mathbb{C}$ *is multiplication by* -2

(3) $(|. \otimes \cap) \circ (\cup \otimes |) = | = (\cap \otimes |) \circ (| \otimes \cup) : V \to V$

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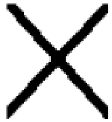
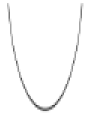
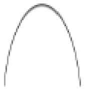
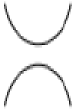
Define the map  $: V \otimes V \to V \otimes V$ by  $(x \otimes y) = y \otimes x$

Consider the composition  $\circ$  $=$  $: V \otimes V \to V \otimes V$

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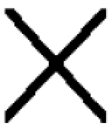
(0)  is an intertwiner

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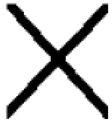
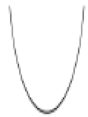
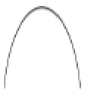
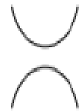
Define the map  $: V \otimes V \to V \otimes V$ by  $(x \otimes y) = y \otimes x$

Consider the composition  $\circ$  $=$  $: V \otimes V \to V \otimes V$

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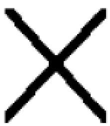
(0)  is an intertwiner

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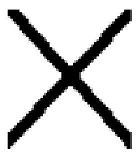
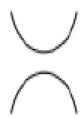
 $= [$  $] + [\ |\ \otimes\ |\ ]$

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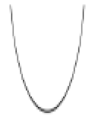
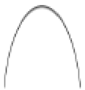
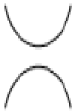
Consider the composition  $\circ$  $=$  $: V \otimes V \to V \otimes V$

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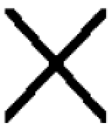
(0)  is an intertwiner

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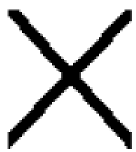
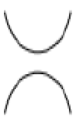
 $= [$  $] + [\,|\, \otimes \,|\,]$

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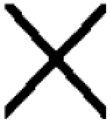
Define the map  $: V \otimes V \to V \otimes V$ by  $(x \otimes y) = y \otimes x$

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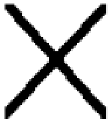
(0)  is an intertwiner

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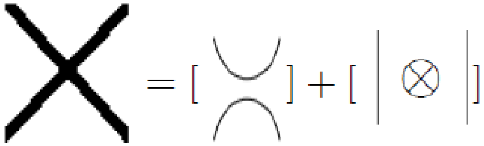
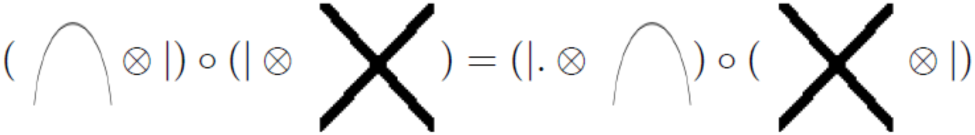
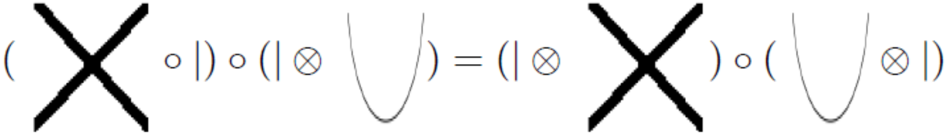


(2) 

(3) 

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Theorem *There is a representation* r_A *of* B_n *on* $V^{\otimes n}$ *defined by*

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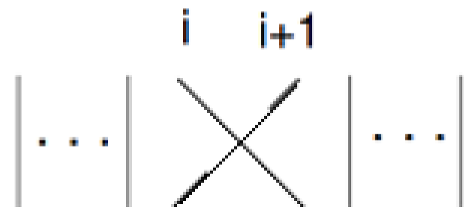

for $i = 1, 2, ..., n-1.$

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$$1, h_1, ...h_n$$

subject to the relations

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$$h_k h_l = h_l h_k \quad \text{for } |k - l| > 1$$

$$h_k h_k = -2h_k \quad \text{for } \textit{all } k$$

$$h_k h_{k\pm 1} h_k = h_k \quad \text{for } \textit{all } \text{ meaningful values of } k$$

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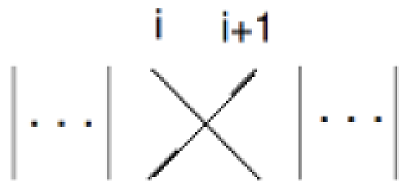

This representation is faithful for all n.

The quantum group $U_q(sl_2)$

Definition *Let* $q \neq 0, 1, -1$ *in* $\mathbb{C}$. *The quantum group* $U_q(sl(2))$ *is the algebra over* $\mathbb{C}$, *with the unit element* 1, *generated by* E, F, K *and* K^{-1}, *subject to the relations*

$$\begin{aligned} K.K^{-1} &= K^{-1}.K = 1 \\ KE &= qEK, \\ KF &= q^{-1}FK \\ EF - FE &= \frac{K^2 - K^{-2}}{q - q^{-1}} \end{aligned}$$

$U_q(sl_2)$-modules

Definition *A* $U_q(sl(2))$*−module is a vector space* V *together with three fixed linear operators* E_V, F_V *and* K_V*, which satisfy*

$$\begin{aligned} K_V.K_V^{-1} &= K_V^{-1}.K_V = 1 \\ K_V E_V &= q E_V K_V \\ K_V F_V &= q^{-1} F_V K_V \\ E_V F_V - F_V E_V &= \frac{K_V^2 - K_V^{-2}}{q - q^{-1}} \end{aligned}$$

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Definition *Given two* $U_q(sl(2))-$*modules* V *and* W*,an intertwiner from* V *to* W *is a linear map* $\phi : V \to W$ *such that*

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The set of all morphisms from V *to* W *will be denoted* $Hom_{U_q(sl(2))}(V, W)$.

$U_q(sl_2)$-modules

For each $n \geq 0$, let V_n the vector space of homogeneous polynomials of degree n in x and y. This space V_n has a basis the monomials $x^n, x^{n-1}y, ..., xy^{n-1}, y^n$ and this basis show that $dim(V_n) = n + 1$.

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$K\ x^{n-j}y^j = A^{n-2j}x^{n-j}y^j \quad (A = q^2)$

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In order to verify that the action of E, F, and K defines a $U_q(sl(2))-$module one must check that the $U_q(sl(2))$ are satisfied.

$U_q(sl_2)$-modules

Definition *Let V be a $U_q(sl(2))-$module and λ be a scaler. A vector $v \neq 0$ in V is said to be of weight $\lambda \in \mathbb{C}$ if $Kv = \lambda v$. If we have, in addition, $Ev = 0$ then v is called a highest weight vector.*

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$U_q(sl_2)$-simple modules

Theorem *Let V_n be the $U_q(sl(2))-$module defined above. The following are satisfied*

(1) V_n is a simple $U_q(sl(2))-$module.

(2) If V is any finite-dimensional simple $U_q(sl(2))-$module then V is isomorphic to V_n for some n.

$U_q(sl_2)$-simple modules

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Some notation..

Let $n \geq 1$ and let $j \in \{n, n-2, ..., -n+2, -n\}$. Let $e_{n,j} := x^{\frac{1}{2}(n+j)} y^{\frac{1}{2}(n-j)}$. In this notation $E\, e_{n,j} = [n-2j] e_{n,j+2}$, $F\, e_{n,j} = [n-2j] e_{n,j-2}$, and $K e_{n,j} = A^j e_{n,j}$

Module structure on tensor product

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One must check that the $U_q(sl(2))$ relations are satisfied.

Theorem *Let $i, j \geq 0$. Then $V_i \otimes V_j = \bigoplus_{|i-j|+1 \leq k \leq i+j-1} V_k$ where the sum runs over all odd $i + j + k$.*

Some morphisms between modules of $U_q(sl_2)$

Let A$\neq 0 \in \mathbb{C}$ be fixed and let $V := V_1 = span\{x, y\}$ be the $U_q(sl(2))-$module defined eariler.

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Define $\cap : V \otimes V \to \mathbb{C}$ via

$$\begin{aligned}
\cap(x \otimes x) &= 0 \\
\cap(x \otimes y) &= iA \\
\cap(y \otimes x) &= -iA^{-1} \\
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\end{aligned}$$

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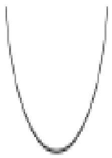
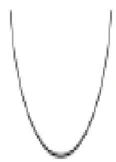
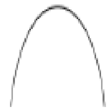
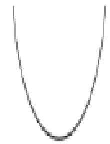
Define  $: \mathbb{C} \to V \otimes V$ by

 $(1) = iAx \otimes y - iA^{-1} y \otimes x$

Some morphisms between modules of $U_q(sl_2)$

Lemma *(1) The maps*  *and*  *are intertwiner operators for* $U_q(sl(2))$.

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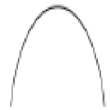
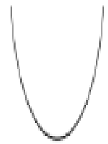
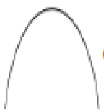
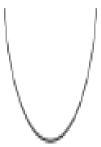
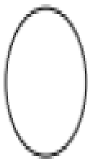
Lemma *(1) The maps*  *and*  *are intertwiner operators for* $U_q(sl(2))$.

(2) The maps  $\circ$  $=$  $: \mathbb{C} \to \mathbb{C}$ *is multiplication by* $-A^2 - A^{-2}$.

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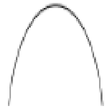
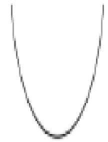
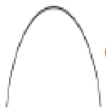
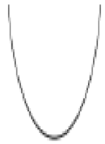
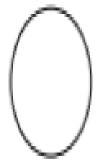
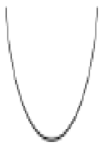
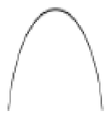
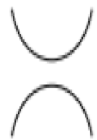
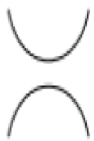
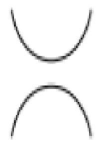
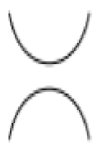
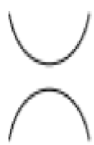
Lemma *(1) The maps*  *and*  *are intertwiner operators for* $U_q(sl(2))$.

(2) The maps  $\circ$  $=$  $: \mathbb{C} \to \mathbb{C}$ *is multiplication by* $-A^2 - A^{-2}$.

(3) The maps  $\circ$  $=$  $: V \otimes V \to V \otimes V$ *is given by*

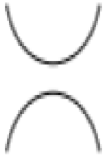
 $(x \otimes x) \quad = \quad$  $(y \otimes y) = 0$

 $(x \otimes y) \quad = \quad -qx \otimes y + y \otimes x$

 $(y \otimes x) \quad = \quad x \otimes y - q^{-1} y \otimes x$

Some morphisms between modules of $U_q(sl_2)$

Define a intertwiner map  $: V \otimes V \to V \otimes V$ as follows

 $:= A[$  $] + A^{-1}[\ \big| \otimes \big|\ ]$

A representation of B_n

Recall the the braid group is given by generators and relations as follows
$B_n =< b_1, ..., b_{n-1} | \ b_i b_j = b_i b_j$ if $|i - j| \geq 2, b_i b_{i+1} b_i = b_{i+1} b_i b_{i+1}$ if $i = 1, 2, ..., n-2 >$

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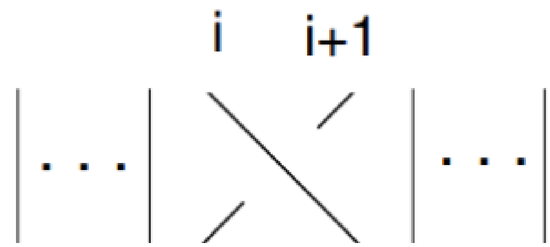
Theorem *There is a representation* r_A *of* B_n *on* $V^{\otimes n}$ *defined by*

$r_A(b_n) =$ 

for $i = 1, 2, ..., n-1.$

The Temperly-Lieb algebra

The Temperley-Lieb Algebra, denoted by $TL_n(\delta)$ where $\delta = -A^2 - A^{-2}$, is the unital $\mathbb{C}[A, A^{-1}]$ algebra of $U_q(sl(2))$ intertwining maps between n-fold tensor powers of the fundamental representation V. In other words

$$TL_n(\delta) = Hom_{U_q(sl(2))}(V^{\otimes n}, V^{\otimes n})$$

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This algebra can be realized as the unital $\mathbb{C}[A, A^{-1}]$ algebra generated by e_i, $1 \leq i \leq n-1$ subject to the relations $e_i e_j = e_j e_i$ if $|i-j| \geq 2$, $e_i e_{i\pm 1} e_i = e_i$, and $e_i^2 = -[2]e_i$.

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Thank You