The tail of a quantum spin network and Andrews-Gordon type identities

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Introduction

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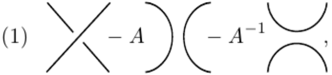
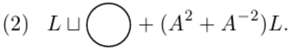
- Armond and Dasbach proved that the tail of an alternating link depends only on the reduced B-graph.
- Garoufalidis and Vuong gave a method to compute the tail of alternating links.

The Kauffman bracket skein module

- The Kauffman Bracket Skein Module of $M = F \times I$ is the $\mathbb{Q}(A)$-free module generated by the set of isotopy classes of framed links in M modulo the submodule generated by the Kauffman relations





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- Relative version can also be defined.

The Jones-Wenzl projector

- We are concerned with the skein module of S^3 and the relative skein module of the disk with $2n$ marked points on the boundary. The latter is usually called the Temperley-Lieb algebra TL_n.

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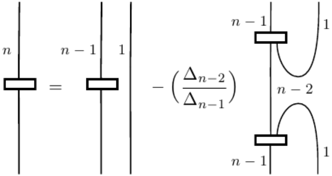
- Define the element $f^{(n)}$ in TL_n via the recursive relation



where

$$\Delta_n = (-1)^n \frac{q^{(n+1)/2} - q^{-(n+1)/2}}{q^{1/2} - q^{-1/2}}.$$

The tail of a sequence of power series

- Write $P_1(q) \doteq_n P_2(q)$,where $P_1(q)$, $P_2(q)$ are two Laurent series if their first n coefficients agree up to sign.

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$$-q^{-4} + 5q^{-3} - 3 + 45q =_5 1 - 5q + 3q^4$$

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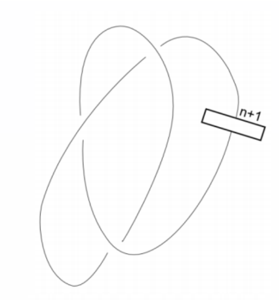
Definition

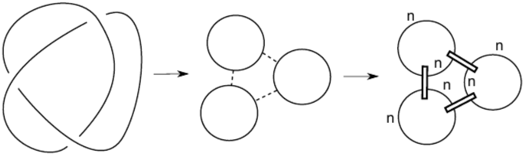
Let $\mathcal{P} = \{P_n(q)\}_{n\in\mathbb{N}}$ be a sequence of formal power series in $\mathbb{Z}[[q]]$. The tail of the sequence $\mathcal{P}$- if it exists - is the formal power series $T_{\mathcal{P}}(q)$ in $\mathbb{Z}[[q]]$ that satisfies

$$T_{\mathcal{P}}(q) \doteq_n P_n(q), \text{ for all } n \in \mathbb{N},$$

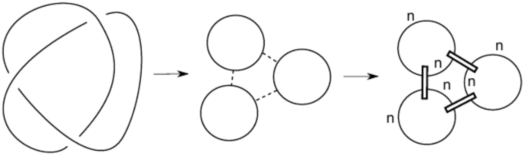
The colored Jones polynomial

- Up to change of variable and multiplication by a power of q the colored Jones polynomial is equal to :



- 

Obtaining $S_B^{(n)}(L)$ for a link L

- 

Obtaining $S_B^{(n)}(L)$ for a link L

- **Theorem (C. Armond)**
  Let L be a link in S^3 and D be a reduced alternating knot diagram of L. Then
  $$\tilde{J}_{n,L}(q) \doteq_{(n+1)} S_B^{(n)}(D).$$

Colored trivalent graph

- A triple of positive integers (a, b, c) is admissible if $a + b + c = 0(mod2)$ and $a + b \geq c \geq |a - b|$.

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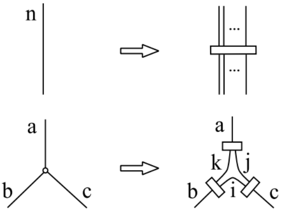
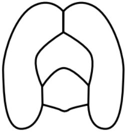
- A colored banded trivalent graph is a trivalent graph with edges labeled by non-negative integers and the three edges meeting at a vertex satisfy the admissibility condition.
- If D is an admissible trivalent graph then the Kauffman bracket evaluation of D is defined by



- Let D be a banded trivalent graph. Let $\mathcal{D} = \{D_n(q)\}_{n\in\mathbb{N}}$ be a sequence of banded trivalent graphs colored admissibly with edges labeled n or $2n$.

e.g.



All edges are colored by $2n$

Ramanujan theta functions

- The general two variable Ramanujan theta function is defined by :

$$f(a,b)=\sum_{i=0}^{\infty} a^{i(i+1)/2} b^{i(i-1)/2}+\sum_{i=1}^{\infty} a^{i(i-1)/2} b^{i(i+1)/2}$$

Ramanujan theta functions

- The second Rogers-Ramanujan identity

$$f(-q^4, -q) = (q, q)_\infty \sum_{i=0}^{\infty} \frac{q^{i^2+i}}{(q, q)_i}$$

Ramanujan theta functions

- The second Rogers-Ramanujan identity

$$f(-q^4,-q)=(q,q)_\infty\sum_{i=0}^{\infty}\frac{q^{i^2+i}}{(q,q)_i}$$

- The Andrews-Gordon identity for the Ramanujan theta function generalizes this identity

$$f(-q^{2k},-q)=(q,q)_\infty\sum_{l_1=0}^{\infty}\sum_{l_2=0}^{\infty}\cdots\sum_{l_{k-1}=0}^{\infty}\frac{q^{\sum_{j=1}^{k-1}(i_j(i_j+1))}}{\prod_{j=1}^{k-1}(q,q)_{l_j}}$$

where $i_j=\sum_{s=j}^{k-1}l_s$.

Ramanujan false theta functions

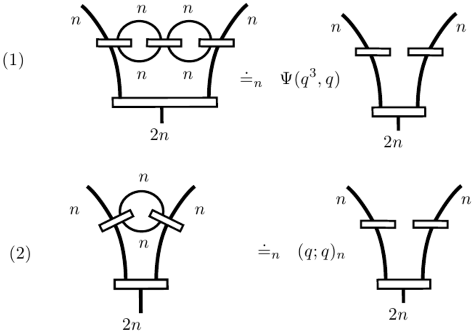
- The general two variable Ramanujan false theta function is given by :

$$\Psi(a,b) = \sum_{i=0}^{\infty} a^{i(i+1)/2} b^{i(i-1)/2} - \sum_{i=1}^{\infty} a^{i(i-1)/2} b^{i(i+1)/2}$$

Ramanujan false theta functions

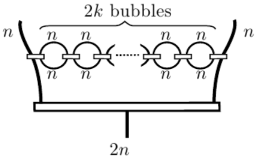
- $$\Psi(q^3, q) = (q, q)_\infty \sum_{i=0}^{\infty} \frac{q^{i^2+i}}{(q;q)_i^2}$$

Theorem (H.)

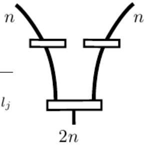


Theorem (H.)

(1)

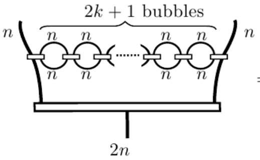


$$\doteq_n \quad (q,q)_n \sum_{l_1=0}^{n} \sum_{l_2=0}^{n} \cdots \sum_{l_k=0}^{n} \frac{q^{\sum_{j=1}^{k}(i_j(i_j+1))}}{(q,q)_{l_k}^2 \prod_{j=1}^{k-1} (q,q)_{l_j}}$$



where $i_j = \sum_{s=j}^{k} l_s$.

(2)



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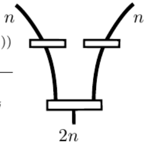

where $i_j = \sum_{s=j}^{k} l_s$.

Theorem (H.)

(1) For all $k \geq 2$:

$$\sum_{i=0}^{\infty} q^{ki^2+(k-1)i} - \sum_{i=1}^{\infty} q^{k(i^2-i)+i} = (q,q)_\infty \sum_{l_1=0}^{\infty}\sum_{l_2=0}^{\infty}\cdots\sum_{l_{k-1}=0}^{\infty} \frac{q^{\sum_{j=1}^{k-1}(i_j(i_j+1))}}{(q,q)_{l_{k-1}}^2 \prod_{j=1}^{k-2}(q,q)_{l_j}}$$

with $i_j = \sum_{s=j}^{k-1} l_s$.

(2) (The Andrews-Gordon identity for the theta function) For all $k \geq 1$

$$\sum_{i=0}^{\infty}(-1)^i q^{k(i^2+i)} q^{i(i-1)/2} + \sum_{i=1}^{\infty}(-1)^i q^{k(i^2-i)} q^{i(i+1)/2} = (q,q)_\infty \sum_{l_1=0}^{\infty}\sum_{l_2=0}^{\infty}\cdots\sum_{l_{k-1}=0}^{\infty} \frac{q^{\sum_{j=1}^{k-1}(i_j(i_j+1))}}{\prod_{j=1}^{k-1}(q,q)_{l_j}}$$

with $i_j = \sum_{s=j}^{k-1} l_s$.

Proof.

- Using linear skein theory one can easily compute the colored Jones polynomial of K_f, the $(2, f)$ torus knot:

$$\tilde{J}_{n,K_f}(q)/\Delta_n(q) = \frac{1}{\Delta_n(q)} \sum_{i=0}^{n} (-1)^{f(n-i)} q^{f(2i+2i^2-2n-n^2)/4} \Delta_{2i}(q).$$

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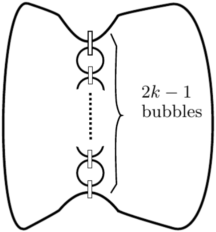
- Hence

$$\tilde{J}_{n,K_{2k}}(q)/\Delta_n(q) \doteq_n \Psi(q^{2k-1}, q)$$

Proof.

- On the other hand

$$\tilde{J}_{n,K_{2k}}(q)/\Delta_n(q) \quad \doteq_n \quad \frac{1}{\Delta_n(q)}$$



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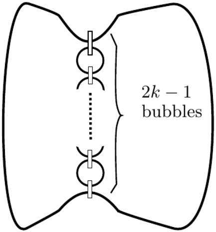
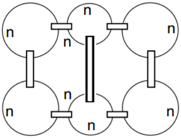


- The tail of the skein element on the right is equal to

$$\tilde{J}_{n,K_{2k}}(q)/\Delta_n(q) \doteq_n (q,q)_n \sum_{l_1=0}^{n} \sum_{l_2=0}^{n} \cdots \sum_{l_{k-1}=0}^{n} \frac{q^{\sum_{j=1}^{k-1}(i_j(i_j+1))}}{(q,q)_{l_k}^2 \prod_{j=1}^{k-2}(q,q)_{l_j}}$$

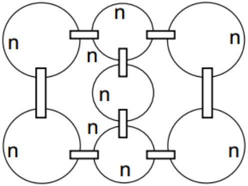
Other identities?



- Using the local skein identities one could compute

$$T_\Gamma = (q;q)_\infty^2 \sum_{i=0}^{\infty} \sum_{j=0}^{\infty} \frac{q^{i+i^2+j+j^2}}{(q;q)_i^2 (q;q)_j^2} = (\Psi(q^3,q))^2$$

Other identities?



The reduced B-graph of 8_5

- **Proposition (H.)**

$$T_{8_5} = (q;q)_\infty^2 \sum_{i=0}^{\infty} \sum_{j=0}^{\infty} \frac{q^{i+i^2+j+j^2}(q;q)_{i+j}}{(q;q)_i^2 (q;q)_j^2}$$

Thank you