SU(2) and SO(3) Representations

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Quaternions

- The quaternions, denoted by H, are defined to be the set of all numbers t=a+bi+cj+dk

  where i,j, and k satisfies i^2=j^2=k^2= -1, ijk= -1.
- H is homoemophirphic to R4.
- H form a non-commutitive division algebra.
- The set of unit quaternions (a^2+b^2+c^2+d^2=1) is a 3 sphere in H. This unit sphere is topologically the same as S3.
- Hence, S3 can be considered a group under quaternion multiplication.
- Like S1, S3 can be used to talk about rotation.

Quaternions

- A quaternion t of absolute value 1 has a real part cos(x) and an “imaginary part” of the absolute value of sin(x), othrogonal to the real part and hence in Ri+Rj+Rk.

$$t=\cos(x)+u\sin(x)$$

where u is a unit vector in Ri+Rj+Rk.

Quaternions

Such a unit quaternion t induces a rotation of $\mathbb{R}\mathbf{i}+\mathbb{R}\mathbf{j}+\mathbb{R}\mathbf{k}$, though not simply by multiplication, since the product of t and a member q of $\mathbb{R}\mathbf{i}+\mathbb{R}\mathbf{j}+\mathbb{R}\mathbf{k}$ may not belong to $\mathbb{R}\mathbf{i}+\mathbb{R}\mathbf{j}+\mathbb{R}\mathbf{k}$. Instead, we send each $q \in \mathbb{R}\mathbf{i}+\mathbb{R}\mathbf{j}+\mathbb{R}\mathbf{k}$ to $t^* qt$, which turns out to be a member of $\mathbb{R}\mathbf{i}+\mathbb{R}\mathbf{j}+\mathbb{R}\mathbf{k}$.

SU2

- A general matrix element of SU2 takes the form

$$U = \begin{pmatrix} \alpha & -\overline{\beta} \\ \beta & \overline{\alpha} \end{pmatrix}$$

- Consider the map

$$\rho : \mathbb{C}^2 \longrightarrow \mathrm{M}(2, \mathbb{C})$$

$$\varphi(\alpha, \beta) = \begin{pmatrix} \alpha & -\overline{\beta} \\ \beta & \overline{\alpha} \end{pmatrix}.$$

SU2

- Consider C2 as R4 and M(C,2) as R8.
- This map is injective real linear map and hence an embedding. Now considering the restriction of on S3, since the domain is compact and since the range is Hausdorff, then this restriction is actually a homeomorphism.

$$\varphi(S^3) = \mathrm{SU}_2(\mathbb{C})$$

SU2

- Topology Structure

1. SU2 is path connected
2. SU2 is compact
3. SU2 is a Lie group

SO(3)

- Set Definition

$$SO(3) = \{A \in GL(n, \mathbb{R}) | AA^T = I, \det(A) = 1\}$$

- Group Structure

1. **It is a group under product of linear transformation**
2. **Any element of SO(3) preserves length orientation.**
3. **It is non-abelian.**
4. **Every element of SO(3) is rotation by some angel θ in [0,π] around some unit vector u.**

SO(3)

- Topology Structure

1. SO(3) has a Lie group structure.
2. The ball with antipodal surface points, identified RP(3) is a smoothmanifold, and this manifold is diffeomorphic to the SO(3).
3. SO(3) is path connected.
4. SO(3) not simply connected. The fundamental group of SO(3) is Z_2.
5. SO(3) is compact.

SO(3) and SU(2)

Consider the map

$$\begin{aligned} f &: H \to H \\ f_q(p) &= qpq^* \end{aligned}$$

Denote by $\mathbb{R}^3$ to the subset $\mathbb{R}i + \mathbb{R}j + \mathbb{R}k \subset H$, and denote by $\mathbb{R}$ to the real part of H. Let u be the unit vector in $\mathbb{R}^3$.

This map is

1- Bijection.

2-Linear.

3-$f|\mathbb{R}$ is the identity.

4-f maps $\mathbb{R}u$ (mulitples of) u to u.

5-f maps $\mathbb{R}^3$ to $\mathbb{R}^3$.

From the properties above, f defines a linear map of $\mathbb{R}^3$that preseves th line in the direcetion of u, Hence, f is a rotation about the vector u. Hence we can consider f as an element of $SO(3)$

SO(3) and SU(2)

We know that we can identify $SU(2)$ with S^3, as lie groups. So, using the map f that we defined above, we can define the following map

$$K : SU(2) \to SO(3)$$

$$K(q) = f_q$$

Note that $f_{q_1 q_2}(p) = (q_1 q_2)p(q_1 q_2)^* = q_1 f_{q_2}(p) q_1^* = f_{q_1}(f_{q_2}(p)) = (f_{q_1} \circ f_{q_2})(p)$ for all p, and hence K is a group homomorphism.

Moreover, if $K(q) = I$, then $qpq^* = I$ for all p in S^3, choose $p = 1$, the $q = \pm I$.

We know that every element of $SO(3)$ is rotation by some angel $\theta \in [0, \pi]$ around some unit vector u. Hence F is a surjective map.

Thus,

$$\frac{SU(2)}{\{\pm I\}} = SO(3)$$

SO(3) and SU(2)

- The map F is the same topologically as the map from S3 to SO(3) that maps x and –x in S3 to one point in SO(3). Hence SO(3) is homeomorphic to RP3.

SO(3)

- Topology Structure

1. The universal cover of SO(3) is a Lie group is isomorphic to the SU(2).
2. The universal cover of SO(3) diffeomorphic to the unit *S*3 and can be understood as the group of unit quaternion.
3. The map from *S*3 onto SO(3) that identifies antipodal points of *S*3 is a surjective homomorphism of Lie groups, with kernal {±1}. Topologically, this map is a two-to-one covering map.

Lie Algebras of SU2 and SO3

Remember that

exp(Lie Algebra)=Lie Group

Lie Algebra of SU2

Exponentiation theorem for $\mathbb{H}$. *When we write an arbitrary element of* $\mathbb{R}\mathbf{i}+\mathbb{R}\mathbf{j}+\mathbb{R}\mathbf{k}$ *in the form* θu, *where* u *is a unit vector, we have*

$$e^{\theta u} = \cos\theta + u\sin\theta$$

and the exponential function maps $\mathbb{R}\mathbf{i}+\mathbb{R}\mathbf{j}+\mathbb{R}\mathbf{k}$ *onto* $\mathbb{S}^3 = \mathrm{SU}(2)$.

Lie Algebra of SU2

- **The space Ri+Rj+Rk mapped onto SU(2) by the exponential function is the *tangent space at* 1 of SU(2), just as the line R*i* is the tangent line at 1 of the circle SO(2).**

- **The three-dimensional space Ri+Rj+Rk is the *tangent space* of the 3-sphere S3 = SU(2) at the identity element.**

Lie Algebra of SU2

- The cross product on R3 is the same as the Lie bracket on Ri+Rj+Rk.

The Lie algebra of SU2 is basically R3 equipped with the cross product structure

Lie Algebra of SU2

By definition $su(2) = \{X \in gl(2,\mathbb{C}) | \exp tX \in SU(2) \text{ for all } t \in \mathbb{R}\}$

$$su(2) = \{X \in gl(2,\mathbb{C}) | trX = 0 and X + X^* = 0\}$$

It can be shown that $\left\{ i = X_1 = \begin{pmatrix} 0 & i \\ -i & 0 \end{pmatrix}, j = X_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, k = X_3 = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix} \right\}$

form a basis of su(2) over the reals. And

$$[X1, X2] = X3, [X2, X3] = X1, [X3, X1] = X2$$

Lie Algebra of SO3

$$so(3):\left\{P=2\begin{pmatrix}0&-1&0\\1&0&0\\0&0&0\end{pmatrix},Q=2\begin{pmatrix}0&0&-1\\0&0&0\\1&0&0\end{pmatrix},R=2\begin{pmatrix}0&0&0\\0&0&-1\\0&1&0\end{pmatrix}\right\}$$

is a basis for so(3)

and [P,Q]=R, [Q,R]=P, [R,P]=Q

Then the map $\phi(xX_1+yX_2+zX_3)=xP+yQ+zR$ for $x,y,z\in\mathbb{R}$

$\phi(xX_1+yX_2+zX_3)=xP+yQ+zR$ for $x,y,z\in\mathbb{R}$ and satisfies $\phi([U,V])=[\phi U,\phi V]$.

Lie Algebra of SO3

- From the map from SU2 to SO3 we can conclude that the tangent space at I of SO(3) is R3.

The Lie algebra of SU2 is basically R3 equipped with the cross product structure

Summary

- The Lie Algebras of SU(2) and SO(3) are isomorphic
- This means that SU(2) and SO(3) are LOCALLY "the same".
- This DOES NOT mean that SU(2) and SO(3) are "the same".
- SU(2) is actually a double cover of SO(3) and there is a 2$\rightarrow$1 surjective Lie group homomorphism from SU(2) to SO(3) .
- How SO(3) representations are related to representations of its double-cover SU(2)?

SU(2) Representation

$SU(2)$ acts on the left space $(\mathbb{C}^2)^* = \{z = (z_1, z_2) | z_1, z_2 \in \mathbb{C}\}$ on the right via the action:

$$z = (z_1, z_2) \to zg = (az_1 + cz_2, bz_1 + dz_2)$$

where $g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SU(2)$.

For each n$\geq$ 0, let H$_n$ be the space of homogeneous polynomial in two complex variables of degree n, i.e,

$$H_n\{f(z_1, z_2) = \sum_{k=0}^{n} \alpha_k z_1^k z_2^{n-k} : \alpha_k \in \mathbb{C}\}$$

SU(2) Representation

H_n is a linear vector space and the set of all polynomials $\phi_k(z_1, z_2) = z_1^k z_2^{n-k}, 0 \leq k \leq n$ form a basis of H_n, hence $dim(H_n) = n + 1$.

Define inner product <> on Hn by

$$< \quad \phi_k, \phi_j >= 0 \quad \text{if} \quad j \neq k$$
$$< \quad \phi_k, \phi_k >= k!(n-k)!$$

Then for $f, h \in H_n, f(z) = \sum_{k=0}^{n} \alpha_k z_1^k z_2^{n-k}, h(z) = \sum_{k=0}^{n} \beta_k z_1^k z_2^{n-k}, < f, h >= \sum_{k=0}^{n} \alpha_k \beta_k^* k!(n-k)!.$

SU(2) Representation

For each $n \geq 0$, let H_n be the Hilbert space defined above. If $g \in SU(2)$, define the map π_n on H_n by

$$\begin{aligned} \pi_n &: SU(2) \rightarrow GL(H_n) \\ \pi_n(g) &= f(z.g), \quad f \in Hn \end{aligned}$$

This map is a representation of $SU(2)$.

SU(2) Representation

To show that this is really defines a representation:

$$[\pi_n(gh)f](z) = f(x\cdot(gh)) = f((x\cdot g)\cdot h) = [\pi_n(h)f](z\cdot g) = [T(g)T(h)f](z)$$

That is $\pi_n(gh) = \pi_n(g)\pi_n(h)$.

Proposition *for each* $n \geq 0$, *the representation* (π_n, H_n) *of* $SU(2)$ *is an irreducible unitary representation.*

SU(2) Representation

To see that (π_n, H_n) is unitary, first consider the subset

$$A = \{\psi_a : \psi_a(z) = (za)^n, a \in \mathbb{C}^2\} \subset H_n$$

Where

$$\psi_a(z) = (za)^n = (a_1 z_1 + a_2 z_2)^n = \sum_k \binom{n}{k} a_1^k a_2^{n-k} z_1^k z_2^{n-k}$$

SU(2) Representation

Then

$$\pi_n(g)\psi_a(z) = \psi_a(zg) = (zga)^n = \psi_{ga}(z)$$

So, for $a, b \in \mathbb{C}^2$

$$< \pi_n(g)\psi_b, \pi_n(g)\psi_b >=< \psi_{ga}, \psi_{gb} >$$

SU(2) Representation

An easy calculation shows that

$$<\psi_a, \psi_b>= n!(a_1 a_2^* + b_1 b_2^*)^n = n!(a, b)^n_{\mathbb{C}^2}$$

It follows that

$$<\pi_n(g)\psi_b, \pi_n(g)\psi_b>=<\psi_{ga}, \psi_{gb}>= n!(ga, gb)^n_{\mathbb{C}^2} = n!(a, b)^n_{\mathbb{C}^2} =<\psi_a, \psi_b>$$

Where $(ga, gb) = (a, b)$ since $g \in SU(2)$is a unitary matrix.

Thus we have proved that $\pi_n(g)$, $g \in SU(2)$ preserves the inner product for elements in the subset A. But the basis for H_n is a subset of A, hence π_n is a unitary representation

SU(2) Representation

To prove that (π_n, H_n) is irreducible, we use Schur's lemma. So we have to show that if $T : H_n \to H_n$ is a linear transformation such that

$$T\pi_n(g) = \pi_n(g)T \qquad \text{for all} \quad SU(2)$$

then $T = \alpha I$.

SU(2) Representation

Any irreducible unitary representation of $SU(2)$ is equivalent to π_n for some $n \in N$.

SU(2) Representation

Let (π, H_π), and (ρ, H_ρ) be two irreducible unitary representation of a compact groupG. Then

(i) If π and ρ are equivalent, $<\chi_\pi, \chi_\rho>_{L(G)} = 1$

(ii)If π and ρ are inequivalent, $<\chi_\pi, \chi_\rho>_{L(G)} = 0$

If $\hat{G}$ is the set of equivalence classes of irreducible unitary representation, then

It follows that the set $\{\chi_\lambda : \lambda \in \hat{G}\}$ is an orthonormal set in $\mathrm{L}^2(G)$

Fourier Analysis on S3

The SU(2) represesentation on $L^2(S^3)$ is isomorphic to the orthogonal sum $\bigoplus_{m\geq 0}(m+1)H_n$

SO(3) Representation

Now consider the irreducible unitary representation π_n of $SU(2)$ given previously

It is easy to see that $\pi_n\ (\pm Id) = Id$ if and only if $n = 0, 2, 4, ...$

We conclude that the irreducible unitary representaion σ_k of $SO(3)$ are indexed by non-negative integers, with

$\tilde{\sigma}_k = \pi_{2k}$, $k = 0, 1, 2, ...$, and $dim \sigma_k = 2k + 1$.

Thank You