Representations of Finite Groups

Vigre Seminar
Mustafa Hajij

Definition

Given a group G and vector space V over a field k, a representation of G on V over the field k is a homomorphism

$$\pi : G \to GL(V)$$

where $GL(V)$, the general linear group on V.

Definition

Given a group G *and vector space* V *over a field* k*, a representation of* G *on* V *over the field* k *is a homomorphism*

$$\pi : G \to GL(V)$$

where $GL(V)$*, the general linear group on* V.

Let V and W be vector spaces over the field k. *Two representations* $\pi_1 : G \to GL(V)$ *and* $\pi_2 : G \to GL(W)$ *are said to be equivalent if there exists a vector space isomorphism* $\alpha : V \to W$ *such that* $\alpha \circ \pi_1(g) \circ \alpha^{-1} = \pi_2(g)$ *for all* g *in* G .

Some representations of cyclic groups

Consider the cyclic group $C_n =< g|g^n = 1 >$. Any homomorphism $\pi : C_n \to GL(1, \mathbb{C}) = \mathbb{C}\backslash\{0\}$ is defined by sending the generator g to non-zero complex number $\pi(g)$ that satisfies $(\pi(g))^n = 1$.

Some representations of cyclic groups

Consider the cyclic group $C_n =< g|g^n = 1 >$. Any homomorphism $\pi : C_n \to GL(1, \mathbb{C}) = \mathbb{C}\backslash\{0\}$ is defined by sending the generator g to non-zero complex number $\pi(g)$ that satisfies $(\pi(g))^n = 1$. So if ξ is an nth primitive root of unity then the representation $\pi : C_n \to S^1 \subset \mathbb{C}\backslash\{0\}$ must belong to the family of representations $\{\pi_j |\ j = 0, ..., n-1\}$ where $\pi_j : C_n \to S^1$ is defined by $\pi_j(g) = \xi^j$.

Some representations of cyclic groups

Consider the cyclic group $C_n =< g|g^n = 1 >$. Any homomorphism $\pi : C_n \to GL(1, \mathbb{C}) = \mathbb{C}\backslash\{0\}$ is defined by sending the generator g to non-zero complex number $\pi(g)$ that satisfies $(\pi(g))^n = 1$. So if ξ is an nth primitive root of unity then the representation $\pi : C_n \to S^1 \subset \mathbb{C}\backslash\{0\}$ must belong to the family of representations $\{\pi_j |\ j = 0, ..., n-1\}$ where $\pi_j : C_n \to S^1$ is defined by $\pi_j(g) = \xi^j$. It is not hard to see that π_j is not equivalent to π_i for $i \neq j$ where $i, j \in \{0, ..., n-1\}$.

Some representations of S3

1. The trivial representation $\rho_1 : S_3 \to GL(1, \mathbb{C})$ defined $\rho_1(g) = Id$

2. The alternating representation, given by the signature of the permutation $\rho_2 : S_3 \to GL(1, \mathbb{C})$ defined $\rho_2(g) = sgn(g)$

3. The standard representation on $V = \{(z_1, z_2, z_3) \mid z_1 + z_2 + z_3 = 0\}$ with $\rho_3((a, b, c))(z_1, z_2, z_3) = (z_a, z_b, z_c)$.

Irreducible representations

Let (π, V) be a representation of a group G over the field k, then we call the subspace W of V a subrepresnetation of V if W is G-stable.

Irreducible representations

Let (π, V) be a representation of a group G over the field k, then we call the subspace W of V a subrepresnetation of V if W is G-stable.

If (π, V) is an arbitrary representation then the zero subspace and V itself are G-stable spaces.

Irreducible representations

Let (π, V) be a representation of a group G over the field k, then we call the subspace W of V a subrepresnetation of V if W is G-stable.

If (π, V) is an arbitrary representation then the zero subspace and V itself are G-stable spaces.

A representaion (π, V) is said to be irreducible if the only G-stable subspaces of V are $\{0\}$ and V itself.

Irreducible representations of a cyclic group

Any one dimensional representation is irreducible.

Irreducible representations of a cyclic group

Any one dimensional representation is irreducible.

Every representation from family of representations $\{\pi_j |\ j = 0, ..., n-1\}$, where $\pi_j : C_n \to S^1$ is defined by $\pi_j(g) = \xi^j$, is irreducible since every one of them is one dimensional.

Irreducible representations of a cyclic group

Any one dimensional representation is irreducible.

Every representation from family of representations $\{\pi_j |\ j = 0, ..., n-1\}$, where $\pi_j : C_n \to S^1$ is defined by $\pi_j(g) = \xi^j$, is irreducible since every one of them is one dimensional.

Theorem The number of inequivalent irreducible representations of a finite group over the complex field $\mathbb{C}$ is equal to the number of distinct conjugate classes of G

Irreducible representations of a cyclic group

Any one dimensional representation is irreducible.

Every representation from family of representations $\{\pi_j |\ j = 0, ..., n-1\}$, where $\pi_j : C_n \to S^1$ is defined by $\pi_j(g) = \xi^j$, is irreducible since every one of them is one dimensional.

Theorem The number of inequivalent irreducible representations of a finite group over the complex field $\mathbb{C}$ is equal to the number of distinct conjugate classes of G

We conclude that the previous family of irreducible representations of the group C_n is the complete list of irreducible representations of this group

Irreducible representations of S_3

The symmetric group S_3 on three letters has three conjugacy classes, represented by the permutations $(1, 2, 3)$, $(2, 1, 3)$, and $(2, 3, 1)$. It also has three irreducible representations; two are one-dimensional and the third is two-dimensional:

Irreducible representations of S_3

The symmetric group S_3 on three letters has three conjugacy classes, represented by the permutations $(1,2,3)$, $(2,1,3)$, and $(2,3,1)$. It also has three irreducible representations; two are one-dimensional and the third is two-dimensional:

1. The trivial representation $\rho_1 : S_3 \to GL(1,\mathbb{C})$ defined $\rho_1(g) = Id$

Irreducible representations of S3

The symmetric group S_3 on three letters has three conjugacy classes, represented by the permutations $(1, 2, 3)$, $(2, 1, 3)$, and $(2, 3, 1)$. It also has three irreducible representations; two are one-dimensional and the third is two-dimensional:

1. The trivial representation $\rho_1 : S_3 \to GL(1, \mathbb{C})$ defined $\rho_1(g) = Id$

2. The alternating representation, given by the signature of the permutation $\rho_2 : S_3 \to GL(1, \mathbb{C})$ defined $\rho_2(g) = sgn(g)$

Irreducible representations of S3

The symmetric group S_3 on three letters has three conjugacy classes, represented by the permutations $(1,2,3)$, $(2,1,3)$, and $(2,3,1)$. It also has three irreducible representations; two are one-dimensional and the third is two-dimensional:

1. The trivial representation $\rho_1 : S_3 \to GL(1,\mathbb{C})$ defined $\rho_1(g) = Id$

2. The alternating representation, given by the signature of the permutation $\rho_2 : S_3 \to GL(1,\mathbb{C})$ defined $\rho_2(g) = sgn(g)$

3. The standard representation on $V = \{(z_1, z_2, z_3) \mid z_1 + z_2 + z_3 = 0\}$ with $\rho_3((a,b,c))(z_1, z_2, z_3) = (z_a, z_b, z_c)$.

Irreducible representations of S3

The symmetric group S_3 on three letters has three conjugacy classes, represented by the permutations $(1,2,3)$, $(2,1,3)$, and $(2,3,1)$. It also has three irreducible representations; two are one-dimensional and the third is two-dimensional:

1. The trivial representation $\rho_1 : S_3 \to GL(1,\mathbb{C})$ defined $\rho_1(g) = Id$

2. The alternating representation, given by the signature of the permutation $\rho_2 : S_3 \to GL(1,\mathbb{C})$ defined $\rho_2(g) = sgn(g)$

3. The standard representation on $V = \{(z_1, z_2, z_3) \mid z_1 + z_2 + z_3 = 0\}$ with $\rho_3((a,b,c))(z_1,z_2,z_3) = (z_a, z_b, z_c)$. This representation is equivalent to the matrix representation

$$\hat{\rho}_3((2,1,3)) = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \quad \hat{\rho}_3((2,3,1)) = \begin{bmatrix} 0 & -1 \\ 1 & -1 \end{bmatrix}$$

Direct Sum of Representations

Let (V, π_1) and (W, π_2) be representations of G. Then $(V \oplus W, \pi_1 \oplus \pi_2)$ where $\pi_1 \oplus \pi_2 : G \to GL(V \oplus W)$ is defined by $\pi_1 \oplus \pi_2(g)(v, w) = (\pi_1(g)(v), \pi_2(g)(w))$, for $g \in G$, $v \in W, w \in W$, is a representation of G called the direct sum of the representations (V, π_1) and (W, π_2).

Direct Sum of Representations

Let (V, π_1) and (W, π_2) be representations of G. Then $(V \oplus W, \pi_1 \oplus \pi_2)$ where $\pi_1 \oplus \pi_2 : G \to GL(V \oplus W)$ is defined by $\pi_1 \oplus \pi_2(g)(v, w) = (\pi_1(g)(v), \pi_2(g)(w))$, for $g \in G$, $v \in W, w \in W$, is a representation of G called the direct sum of the representations (V, π_1) and (W, π_2).

In the case where V and W are finite dimensional of dimension n and m receptively then if we choose a basis $\{v_1, .., v_n\}$ for V and a basis $\{w_1, .., w_n\}$ for W then $\{v_1, .., v_n, w_1, .., w_n\}$ is a basis for $V \oplus W$ and we can use this basis to identify $GL(V \oplus W)$ with $GL(n + m, k)$ and obtain a matrix representation $\pi_1 \oplus \pi_2$

$$\begin{pmatrix} \pi_1(g) & 0 \\ 0 & \pi_2(g) \end{pmatrix}$$

Maschke's Theorem

Let G be a finite group and (π, V) be a nonzero finite dimensional representation of G. Then

$$V = W_1 \oplus \ldots \oplus W_k$$

where W_i is an irreducible subrepresentation of V.

Maschke's Theorem

Let G be a finite group and (π, V) be a nonzero finite dimensional representation of G. Then

$$V = W_1 \oplus \ldots \oplus W_k$$

where W_i is an irreducible subrepresentation of V.

Corollary *Let G be a finite group and let (π, V) be a representation of G of dimension d. Then there is a fixed invertable matrix T such that every matrix $\pi(g)$, $g \in G$, has the form*

$$T\pi(g)T^{-1} = \begin{pmatrix} \pi_1(g) & 0 & \ldots & 0 \\ 0 & \pi_2(g) & \ldots & 0 \\ . & . & . & . \\ . & . & . & . \\ . & . & . & . \\ 0 & 0 & \ldots & \pi_k(g) \end{pmatrix}$$

where each π_i is an irreducible matrix representation of G.

Tensor product of representations

Let (V, π_1) and (W, π_2) be representations of G. Then $(V \otimes W, \pi_1 \otimes \pi_2)$, where $\pi_1 \otimes \pi_2 : G \to GL(V \otimes W)$ is defined by $(\pi_1 \otimes \pi_2)(g) = \pi_1(g) \otimes \pi_2(g)$, for all g in G, is a representation of G called the tensor product of the representations (V, π_1) and (W, π_2).

Tensor product of representations

Let (V, π_1) and (W, π_2) be representations of G. Then $(V \otimes W, \pi_1 \otimes \pi_2)$, where $\pi_1 \otimes \pi_2 : G \to GL(V \otimes W)$ is defined by $(\pi_1 \otimes \pi_2)(g) = \pi_1(g) \otimes \pi_2(g)$, for all g in G, is a representation of G called the tensor product of the representations (V, π_1) and (W, π_2).

Theorem Let G and H be groups.

1. If π and ρ are irreducible representations of G and H, repectively, then $\pi \otimes \rho$ is an irreducible representation of $G \times H$.

2. If π_i and ρ_j are complete list of inequivalent irreducible representations for G and H, repectively, then $\pi_i \otimes \rho_j$ is a complete list of inequivalent irreducible representations for $G \times H$.

Tensor product of representations

Let (V, π_1) and (W, π_2) be representations of G. Then $(V \otimes W, \pi_1 \otimes \pi_2)$, where $\pi_1 \otimes \pi_2 : G \to GL(V \otimes W)$ is defined by $(\pi_1 \otimes \pi_2)(g) = \pi_1(g) \otimes \pi_2(g)$, for all g in G, is a representation of G called the tensor product of the representations (V, π_1) and (W, π_2).

Theorem Let G and H be groups.

1. If π and ρ are irreducible representations of G and H, repectively, then $\pi \otimes \rho$ is an irreducible representation of $G \times H$.

2. If π_i and ρ_j are complete list of inequivalent irreducible representations for G and H, repectively, then $\pi_i \otimes \rho_j$ is a complete list of inequivalent irreducible representations for $G \times H$.

We can use this theorem to write the complete list of inequivalent irreducible representations for any finite abelian group

Group Characters

Let (V, π) be a representation of G. The character of π is the function $\chi_\pi : G \to \mathbb{C}$ defined by $\chi_\pi(g) = Tr(\pi(g))$ for all g in G.

Group Characters

Let (V, π) be a representation of G. The character of π is the function $\chi_\pi : G \to \mathbb{C}$ defined by $\chi_\pi(g) = Tr(\pi(g))$ for all g in G.

Note that if (V, π_1) and (W, π_2) are two equivalent represntation of G then there exists a vector space isomorephism $T : V \to W$ such that $\alpha \circ \pi_1(g) \circ \alpha^{-1} = \pi_2(g)$ and hence

$$\chi_{\pi_2}(g) = Tr(\pi_2(g)) = Tr(\alpha \circ \pi_1(g) \circ \alpha^{-1}) = Tr(\pi_1(g)) = \chi_{\pi_1}(g)$$

Group Characters

Let (V, π) be a representation of G. The character of π is the function $\chi_\pi : G \to \mathbb{C}$ defined by $\chi_\pi(g) = Tr(\pi(g))$ for all g in G.

Note that if (V, π_1) and (W, π_2) are two equivalent represntation of G then there exists a vector space isomorephism $T : V \to W$ such that $\alpha \circ \pi_1(g) \circ \alpha^{-1} = \pi_2(g)$ and hence

$$\chi_{\pi_2}(g) = Tr(\pi_2(g)) = Tr(\alpha \circ \pi_1(g) \circ \alpha^{-1}) = Tr(\pi_1(g)) = \chi_{\pi 1}(g)$$

For a matrix representation of dimension n,

$$\chi_\pi \ (g) = \sum_{i=1}^{n} (\pi(g))_{ii}$$

Group Characters

Let (V, π) be a representation of G. The character of π is the function $\chi_\pi : G \to \mathbb{C}$ defined by $\chi_\pi(g) = Tr(\pi(g))$ for all g in G.

Note that if (V, π_1) and (W, π_2) are two equivalent represntation of G then there exists a vector space isomorephism $T : V \to W$ such that $\alpha \circ \pi_1(g) \circ \alpha^{-1} = \pi_2(g)$ and hence

$$\chi_{\pi_2}(g) = Tr(\pi_2(g)) = Tr(\alpha \circ \pi_1(g) \circ \alpha^{-1}) = Tr(\pi_1(g)) = \chi_{\pi 1}(g)$$

For a matrix representation of dimension n,

$$\chi_\pi \ (g) = \sum_{i=1}^{n} (\pi(g))_{ii}$$

If the representation π is irreducible then the character χ_π is called an irreducible character

Elementary properties of group characters

(1) $\chi_\pi(e) = \dim \pi$

Elementary properties of group characters

(1) $\chi_\pi(e) = \dim \pi$

(2) $\chi_\pi(g^{-1}) = \overline{\chi_\pi(g)}$ for every g in G

Elementary properties of group characters

(1) $\chi_\pi(e) = \dim \pi$

(2) $\chi_\pi(g^{-1}) = \overline{\chi_\pi(g)}$ for every g in G

(3) $\chi_{\pi_1 \oplus \pi_2} = \chi_{\pi_1} + \chi_{\pi_2}$

Elementary properties of group characters

(1) $\chi_\pi(e) = \dim \pi$

(2) $\chi_\pi(g^{-1}) = \overline{\chi_\pi(g)}$ for every g in G

(3) $\chi_{\pi_1 \oplus \pi_2} = \chi_{\pi_1} + \chi_{\pi_2}$

(4) $\chi_{\pi_1 \otimes \pi_2} = \chi_{\pi_1} \chi_{\pi_2}$

Inner product of characters

Denote by $L^2(G)$ the vector space of functions on G taking values in $\mathbb{C}$. On $L^2(G)$, we can define an inner product by

$$(f_1|f_2) = \frac{1}{|G|} \sum_{g \in G} \overline{f_1(g)} f_2(g)$$

Inner product of characters

Denote by $L^2(G)$ the vector space of functions on G taking values in $\mathbb{C}$. On $L^2(G)$, we can define an inner product by

$$(f_1|f_2) = \frac{1}{|G|}\sum_{g\in G}\overline{f_1(g)}f_2(g)$$

With respect to this inner product the irreducible characters of a finite group over the complex field $\mathbb{C}$ from an orthonormal system. In other words, if (V,π_1) and (W,π_2) are two irreducible representations of a group G. Then

$$(\chi_{\pi_1}|\chi_{\pi_2}) = \delta_{\pi_1,\pi_2}$$

Inner product of characters

Let (π, V) be a representation of a group G over the field $\mathbb{C}$ with character χ_π. Suppose

$$\pi = m_1\pi_1 \oplus m_2\pi_2 \oplus \dots \oplus m_k\pi_k$$

where the π_j are pairwise inequivalent irreducibles with characters χ_{π_i}.

1. $\chi_\pi = m_1\chi_{\pi_1} + m_2\chi_{\pi_2} + \dots + m_k\chi_{\pi_k}$.
2. $(\chi_\pi | \chi_{\pi_j}) = m_j$ for all j.
3. $(\chi_\pi | \chi_\pi) = \sum_{j=1}^{k} m_j^2$.
4. π is irreducible if and only if $(\chi_\pi | \chi_\pi) = 1$.
5. Let ρ be another representation of G with character χ_ρ. Then $\pi \simeq \rho$ if and only if $\chi_\pi(g) = \chi_\rho(g)$ for all g in G.

Inner product of characters

Proof

(1) $\chi_\pi(g) = tr(\pi(g)) = tr((\oplus_{j=1}^{k} m_j \pi_j)(g)) = tr((\oplus_{j=1}^{k} m_j \pi_j(g))) = \sum_{j=1}^{k} (m_j tr(\pi_j(g))) = \sum_{j=1}^{k} m_j \chi_{\pi_j}(g)$ for all g in G.

Inner product of characters

Proof

(1) $\chi_\pi(g) = tr(\pi(g)) = tr((\oplus_{j=1}^k m_j \pi_j)(g)) = tr((\oplus_{j=1}^k m_j \pi_j(g))) = \sum_{j=1}^k (m_j tr(\pi_j(g))) = \sum_{j=1}^k m_j \chi_{\pi_j}(g)$ for all g in G.

(2) $(\chi_\pi | \chi_{\pi_j}) = (\sum_{i=1}^k m_j \chi_{\pi_i} | \chi_{\pi_j}) = \sum_{i=1}^k m_j (\chi_{\pi_i} | \chi_{\pi_j}) = m_j$

Inner product of characters

Proof

(1) $\chi_\pi(g) = tr(\pi(g)) = tr((\oplus_{j=1}^k m_j\pi_j)(g)) = tr((\oplus_{j=1}^k m_j\pi_j(g))) = \sum_{j=1}^k (m_j tr(\pi_j(g))) = \sum_{j=1}^k m_j\chi_{\pi_j}(g)$ for all g in G.

(2) $(\chi_\pi|\chi_{\pi_j}) = (\sum_{i=1}^k m_j\chi_{\pi_i}|\chi_{\pi_j}) = \sum_{i=1}^k m_j(\chi_{\pi_i}|\chi_{\pi_j}) = m_j$

(3) $(\chi_\pi|\chi_\pi) = (\sum_{i=1}^k m_i\chi_{\pi_i}|\sum_{j=1}^k m_j\chi_{\pi j}) = \sum_{i=1}^k\sum_{j=1}^k m_i m_j(\chi_{\pi_i}|\chi_{\pi_j}) = \sum_{i=1}^k m_i^2$

Inner product of characters

Proof

(1) $\chi_\pi(g) = tr(\pi(g)) = tr((\oplus_{j=1}^k m_j\pi_j)(g)) = tr((\oplus_{j=1}^k m_j\pi_j(g))) = \sum_{j=1}^k (m_j tr(\pi_j(g))) = \sum_{j=1}^k m_j \chi_{\pi_j}(g)$ for all g in G.

(2) $(\chi_\pi|\chi_{\pi_j}) = (\sum_{i=1}^k m_j\chi_{\pi_i}|\chi_{\pi_j}) = \sum_{i=1}^k m_j(\chi_{\pi_i}|\chi_{\pi_j}) = m_j$

(3) $(\chi_\pi|\chi_\pi) = (\sum_{i=1}^k m_i\chi_{\pi_i}|\sum_{j=1}^k m_j\chi_{\pi j}) = \sum_{i=1}^k\sum_{j=1}^k m_i m_j(\chi_{\pi_i}|\chi_{\pi_j}) = \sum_{i=1}^k m_i^2$

(4) If $(\chi_\pi|\chi_\pi) = \sum_{j=1}^k m_j^2 = 1$ then there must be exactly one index j such that $m_j = 1$ and all the rest of m_i must be zero. But then $\pi = \pi_j$ which is irreducible by assumption.

Inner product of characters

Proof

(1) $\chi_\pi(g) = tr(\pi(g)) = tr((\oplus_{j=1}^k m_j\pi_j)(g)) = tr((\oplus_{j=1}^k m_j\pi_j(g))) = \sum_{j=1}^k (m_j tr(\pi_j(g))) = \sum_{j=1}^k m_j\chi_{\pi_j}(g)$ for all g in G.

(2) $(\chi_\pi|\chi_{\pi_j}) = (\sum_{i=1}^k m_j\chi_{\pi_i}|\chi_{\pi_j}) = \sum_{i=1}^k m_j(\chi_{\pi_i}|\chi_{\pi_j}) = m_j$

(3) $(\chi_\pi|\chi_\pi) = (\sum_{i=1}^k m_i\chi_{\pi_i}|\sum_{j=1}^k m_j\chi_{\pi j}) = \sum_{i=1}^k\sum_{j=1}^k m_i m_j(\chi_{\pi_i}|\chi_{\pi_j}) = \sum_{i=1}^k m_i^2$

(4) If $(\chi_\pi|\chi_\pi) = \sum_{j=1}^k m_j^2 = 1$ then there must be exactly one index j such that $m_j = 1$ and all the rest of m_i must be zero. But then $\pi = \pi_j$ which is irreducible by assumption.

(5) We can always assume that the expansions of ρ and π contain the the same irreducibles. Suppose that $\rho = n_1\pi_1 \oplus n_2\pi_2 \oplus ... \oplus n_k\pi_k$. Since $\chi_\pi = \chi_\rho$, then $m_j = (\chi_\pi|\chi_{\pi_j}) = (\chi_\rho|\chi_{\pi_j}) = n_j$ for all j. Thus $\pi \simeq \rho$.

Inner product of characters

The previous theorem can be used to

(1) Decomposing an unknown character as a linear combination of irreducible characters.

(2) Constructing the complete character table when only some of the irreducible characters are known.

(3) Finding the order of the group.

Character Table

If ξ is a 4th primitive root of unity then the family of representations $\Omega = \{\pi_j |\ j = 0, 1, 2, 3\ \}$, where $\pi_j : C_4 \to S^1$ is defined by $\pi_j(g) = \xi^j$, is a complete list of irreducible representations.

Character Table

If ξ is a 4th primitive root of unity then the family of representations $\Omega = \{\pi_j |\ j = 0, 1, 2, 3\ \}$, where $\pi_j : C_4 \to S^1$ is defined by $\pi_j(g) = \xi^j$, is a complete list of irreducible representations.

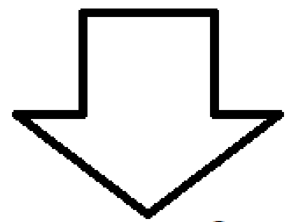
	e	g	g^2	g^3
π_0				
π_1				
π_2				
π_3				

The character table for C_4

Character Table

If ξ is a 4th primitive root of unity then the family of representations $\Omega = \{\pi_j | \ j = 0, 1, 2, 3 \ \}$, where $\pi_j : C_4 \to S^1$ is defined by $\pi_j(g) = \xi^j$, is a complete list of irreducible representations.

Conjugacy classes of the group C_4



	e	g	g^2	g^3
π_0				
π_1				
π_2				
π_3				

The character table for C_4

Character Table

If ξ is a 4th primitive root of unity then the family of representations $\Omega = \{\pi_j |\ j = 0, 1, 2, 3\ \}$, where $\pi_j : C_4 \to S^1$ is defined by $\pi_j(g) = \xi^j$, is a complete list of irreducible representations.

Conjugacy classes of the group C_4

Irreducible representations of C_4

	e	g	g^2	g^3
π_0				
π_1				
π_2				
π_3				

The character table for C_4

Character Table

If ξ is a 4th primitive root of unity then the family of representations $\Omega = \{\pi_j |\ j = 0, 1, 2, 3\ \}$, where $\pi_j : C_4 \to S^1$ is defined by $\pi_j(g) = \xi^j$, is a complete list of irreducible representations.

Conjugacy classes of the group C_4

Irreducible representations of C_4

	e	g	g^2	g^3
π_0	1	1	1	1
π_1				
π_2				
π_3				

The character table for C_4

Character Table

If ξ is a 4th primitive root of unity then the family of representations $\Omega = \{\pi_j |\ j = 0, 1, 2, 3\ \}$, where $\pi_j : C_4 \to S^1$ is defined by $\pi_j(g) = \xi^j$, is a complete list of irreducible representations.

Conjugacy classes of the group C_4

Irreducible representations of C_4

	e	g	g^2	g^3
π_0	1	1	1	1
π_1	1	i	-1	$-i$
π_2	1	-1	1	-1
π_3	1	$-i$	-1	i

The character table for C_4

The Character Table for S3

The symmetric group S_3 on three letters has three conjugacy classes, represented by the permutations $(1,2,3)$, $(2,1,3)$, and $(2,3,1)$. It also has three irreducible representations; two are one-dimensional and the third is two-dimensional:

1. The trivial representation $\rho_1 : S_3 \to GL(1,\mathbb{C})$ defined $\rho_1(g) = Id$

2. The alternating representation, given by the signature of the permutation $\rho_2 : S_3 \to GL(1,\mathbb{C})$ defined $\rho_2(g) = sgn(g)$

3. The standard representation on $V = \{(z_1, z_2, z_3) \mid z_1 + z_2 + z_3 = 0\}$ with $\rho_3(\{a,b,c\})(z_1,z_2,z_3) = (z_a, z_b, z_c)$. This representation is equivalent to the matrix representation

$$\hat{\rho}_3(\{2,1,3\}) = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \quad \hat{\rho}_3(\{2,3,1\}) = \begin{bmatrix} 0 & -1 \\ 1 & -1 \end{bmatrix}$$

The Character Table for S3

So the character table for this representation is

	K_1	K_2	K_3
ρ_1	1	1	1
ρ_2	1	-1	1
$\hat{\rho}_3$	2	0	-1

Thank You